Noncommutative Composite Water-Filling for Energy Harvesting and Smart Power Grid Hybrid System With Peak Power Constraints

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***Abstract*—Energy harvesting makes use of energy from the environment. However, since harvesting energy depends on natural conditions, it is not a stable energy source. As a result, the energy from the power grid is often included to serve as a supplementary source to regulate the overall energy supply of the system. Further, the power from the power grid is often subject to the constraints of peak power and the energy budget. These constraints lead to more difficulties in solving optimal power allocation problems. In this paper, we extend our recently proposed geometric water-filling (GWF) and recursive GWF (RGWF) algorithms to solve the throughput maximization problem and transmission completion time minimization problems for this kind of hybrid energy source system. Our investigation shows that the optimal power allocation for throughput maximization is the result of a sequence of water-filling algorithms for smart power grid and harvested energy, in that order, followed by a power adjustment step of the power from the grid. The allocation order is not commutative for an optimal solution due to the specific structure of the target problems. The proposed algorithms can compute the exact (optimal) solutions to the problems via finite computation with low computational complexity. Numerical examples are presented to illustrate the detailed procedures to efficiently obtain the optimal power allocation solutions using the proposed algorithms. The results also illustrate that the composite operation of the two water-fillings is noncommutative.**

***Index Terms*—Energy harvesting, optimal dynamic power allocation, optimization theory and methods, peak power constraints, smart power grid, water-filling algorithm with mixed constraints.**

I. Introduction

Wireless devices are normally powered by batteries, which need to be either replaced or recharged periodically. One possible technique to overcome this limitation is to harvest energy from the environment, such as vibration absorption devices, solar energy, wind energy, thermal energy, and other clean energy [1]. In such systems, energy harvesting has become a preferred choice for supporting "green communication." The system is normally modeled as a sequence of epochs, where for each epoch, an event occurs that may be the transition consequence between transmitting signal packages, with channel fading gain variation or new energy being harvested, or both.

A. Related Work, Without Peak Power Constraints

This system setting aforementioned leads to new design challenges and insights in a wireless link with a rechargeable transmitter and fading channels [2]–[16]. As fundamental work for transmission with energy harvesting, in [2] and [5], the throughput maximization problem with full side information was investigated. Some approaches were proposed making use of the water-filling algorithm to solve the Karush–Kuhn–Tucker (KKT) conditions [17] of the target problem. For the fading channel cases, [2] presented "directional water-filling scheme," which is an innovative application of water-filling mechanism for green communications. Other effective methods were also presented in [5].

Different from these studies, we recursively applied the proposed geometric water-filling (GWF) [18] to solve the throughput maximization problem and transmission completion time minimization problem in our recent paper for a single antenna [19] and multiple antennas [20].

Since energy harvesting depends on natural conditions, which is not a stable energy source, the energy from the smart power grids is often needed to be considered a supplementary source to regulate the overall energy flow of the system. Optimal power allocation to maximize system throughput turns out to be more complicated in this kind of a hybrid energy source system. Recently, [21] the issues of power allocation problems to minimize the grid power consumption with random energy and data arrival have been investigated, and the structure of the optimal power allocation policy in some special cases has been analyzed.

With water-filling, more power is allocated to the channels with higher gains to maximize the sum of data rates of all the subchannels [22]. The conventional way to solve the water-filling problem is to solve the KKT conditions and then find the water level(s) and the solutions. In [18], we proposed an approach from simple geometric meaning of water-filling (GWF). Optimality of the constructed solution is strictly proven. Due to complexity of solving the KKT conditions of the problem, the GWF is easier to compute than the conventional water-filling and reveals more useful information with less computation. This advantage becomes more significant when the system becomes more complicated, where the GWF is often utilized recursively or iteratively to solve the target problem.

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B. Our Work, Considering Peak Power Constraints

Often, the powers from the power grid are subject to the constraints of peak power and the energy budget. Thus, the power from the power grid can also be controlled. These constraints, which might seem to be a trivial case of an optimization problem, introduce more difficulties in solving the optimal power allocation problems.

In this paper, both GWF with peak power (GWFPP) in [18] to solve the problems with peak power constraints and recursive GWF (RGWF) in [19] to solve power allocation with harvesting energy are utilized, to compute the exact (optimal) solution to the maximum throughput and the minimum transmission completion time problems with peak power constraints from the power grid. GWFPP and RGWF are applied in order, followed by a one-step adjustment to obtain the optimal solution of the hybrid system to maximize system throughput. The composite water-filling algorithm is referred to as hybrid power allocation algorithm 1 (HPA1). It is shown that these two water-filling steps are not commutative. Then, HPA1 is developed to solve the transmission completion time minimization problem. Both discrete-time case (to find the index n of the epoch to complete the transmission) and the continuous-time case (to find time t to complete the transmission) are investigated. The corresponding algorithms are referred to as HPA2 and HPA3, respectively. By checking if the target B bits transmission is completed, the algorithms are constructed and the optimality proof is provided. A conceptual descriptive algorithm to solve the transmission completion time minimization problem was proposed in [2]. Since the proposed optimal power allocation policy mainly results from the recursive computing epoch by epoch, it does not always need the information/solution of the entire process to solve the minimum transmission completion time problem. This is a distinct feature of the proposed algorithms compared with the algorithms reported in the open literature.

According to the definition of online algorithms [23], RGWF in the proposed algorithms possesses some characteristics of an online algorithm. This is because the family of RGWF is defined by recursion, without extrapolation, and its input and computation choose the way of piece-by-piece information in a serial fashion. Furthermore, since the power from the power grid is, in practice, subject to the peak power constraints, these constraints are considered with a more general form in this paper.

In the remainder of this paper, the system model and problem statement are presented in Section II. HPA1 for maximum throughput is investigated in Section III The algorithms of HPA2 and HPA3 are further investigated in Section IV, respectively. Numerical examples and computational complexity discussion are presented in Section V. Section VI concludes this paper.

II. System Model and Problem Statement

Here, the energy harvesting and smart power grid coexisting system model in a fading channel is presented, followed by the optimization problem to maximize the throughput. For convenience and without loss of generality, the process is assumed to be a discrete-time process.

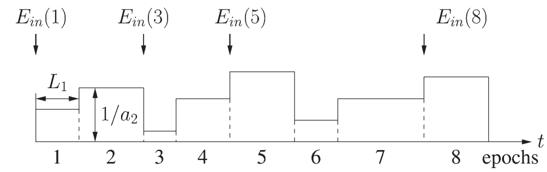

Fig. 1. Epochs and energy arrivals. $K = 8$ epochs in $(0, T]$.

As shown in Fig. 1, the system model shows the time period from $(0, T]$, including K epochs. Let L_i and a_i denote the time duration and the fading power gain of the ith epoch, where $i = 1, \ldots, K$. Without loss of generality, assume $L_i > 0, a_i > 0 \,\forall i$. At the beginning of the ith epoch, the energy arrival is denoted by $E_{\text{in}}(i)$ with $E_{\text{in}}(i) \geq 0$. In addition to the harvested energy $E_{\text{in}}(i)$, the transmission is also connected with the smart power grid. Let $E_{(G,\text{total})}$ denote the energy budget of total energy supported by the power grid. Moreover, we assume $E_{G,i} \geq 0$ as the peak power constraint from the power grid for the ith epoch $\forall i$.

For the hybrid energy source systems, assume that the optimal power management strategy is such that the transmit power is constant in each epoch. Therefore, let us denote the transmit power at epoch i by s_i $(i = 1, \ldots, K)$, which consists of the power from the harvested energy $s_{H,i}$, and the power from the smart power grid $s_{G,i}$. The objective is to maximize the total throughput by the deadline T, i.e., within the K epochs. We have causal constraints due to energy arrivals. The energy capacity of the battery $E_{\max} \gg 0$ is assumed in this paper. With this relaxation, we can compute optimal transmission solutions and can obtain more insights of the problems. The investigation also lays down the foundation to solve the cases of finite $E_{\max}$ and then to carry out real-time computation. Further, the exact (optimal) solution in this paper means an actual optimal solution, which is not a point approximation to the optimal solution. The optimization problem in this hybrid system with fading channels can be written as

$$
\begin{aligned}
\max_{\{s_{H,i}, s_{G,i}\}_{i=1}^{K}} \quad & \sum_{i=1}^{K} \frac{L_i}{2} \log\left(1 + a_i(s_{H,i} + s_{G,i})\right) \\
\text{Subject to} \quad & 0 \leq s_{H,i} \quad \forall i \\
& 0 \leq s_{G,i} \leq E_{G,i} \quad \forall i \\
& \sum_{i=1}^{l} L_i s_{H,i} \leq \sum_{i=1}^{l} E_{\text{in}}(i) \\
& \quad \text{for } l = 1, \ldots, K \\
& \sum_{i=1}^{K} L_i s_{G,i} \leq E_{(G,\text{total})} \qquad (1)
\end{aligned}
$$

where the preceding two constraints account for the nonnegative harvested power, the nonnegative power of the power grid, and the peak power constraints from the power grid; the third constraint accounts for the causal requirement; and the fourth constraint reflects the maximal energy available from the smart power grid.

By interpreting the observed properties of the optimal harvested power allocation as a water-filling scheme, $E_{\text{in}}(i)$ units of water are filled into a rectangle container with bottom width $(L_i/2)\ \forall i$. The last weighted power sum constraint from energy harvesting forms an equality. Furthermore, for unifying parameter notation, through a change of variables, we can obtain an equivalent target problem as follows:

$$\begin{aligned}
\max_{\{s_{H,i}, s_{G,i}\}_{i=1}^K} \quad & \sum_{i=1}^{K} w_i \log\left(1 + a_i(s_{H,i} + s_{G,i})\right) \\
\text{subject to} \quad & 0 \le s_{H,i} \quad \forall i \\
& 0 \le s_{G,i} \le E_{G,i} \quad \forall i \\
& \sum_{i=1}^{l} s_{H,i} \le \sum_{i=1}^{l} E_{\text{in}}(i) \quad \forall l \\
& \sum_{i=1}^{K} s_{G,i} \le E_{(G,\text{total})}
\end{aligned} \tag{2}$$

where $w_i \leftarrow (L_i/2), a_i \leftarrow (a_i/L_i),\ s_{H,i} \leftarrow L_i s_{H,i},\ s_{G,i} \leftarrow L_i s_{G,i}$ and $E_{G,i} \leftarrow L_i E_{G,i}$, for any i. Note that the symbol $\leftarrow$ is the assignment operator. Without consideration of trivial cases, $E_{(G,\text{total})} > 0$ can be assumed. The second constraint remarkably increases difficulty to solve (2).

To find the solution to problem (2), the conventional water-filling approach starts by obtaining the KKT conditions of problem (2) as a set of optimality conditions, and then it solves the conditions to determine the variables $\{s_{H,i}, s_{G,i}\}$ and their dual variables (coefficients), i.e.,

$$\begin{cases}
s_{H,i} + s_{G,i} = \left(\frac{w_i}{\sum_{k=i}^{K} \lambda_k} - \frac{1}{a_i}\right)^+ = \left(\frac{w_i}{\mu + \nu_i} - \frac{1}{a_i}\right)^+ \\
\quad \text{for } i = 1, \ldots, K \\
0 \le s_{H,i} \quad \forall i \\
0 \le s_{G,i} \le E_{G,i} \\
\nu_i(s_{G,i} - E_{G,i}) = 0, \nu_i \ge 0 \quad \forall i \\
\lambda_l \left(\sum_{i=1}^{l} s_{H,i} - \sum_{i=1}^{l} E_{\text{in}}(i)\right) = 0, \quad \lambda_l \ge 0 \\
\sum_{i=1}^{l} s_{H,i} \le \sum_{i=1}^{l} E_{\text{in}}(i), \quad 1 \le l \le K \\
\mu \left(\sum_{i=1}^{K} s_{G,i} - E_{(G,\text{total})}\right) = 0, \quad \mu \ge 0 \\
\sum_{i=1}^{K} s_{G,i} \le E_{(G,\text{total})}
\end{cases} \tag{3}$$

where the function $(x)^+$ means $(x)^+ = x$, for $x \ge 0$, and $(x)^+ = 0$, for $x < 0$. Furthermore, ν_i is the dual variable corresponding to the constraint: $s_{G,i} \le E_{G,i}$, for any i; λ_l is the dual variable corresponding to the lth harvested power sum constraint, for any l; and μ is the dual variable corresponding to the total power sum constraint from the smart power grid. However, by only observing or using the monotonicity information $1/(\sum_{j=i}^{K} \lambda_j)$ with respect to i in the first KKT condition related to the sums of pairs $\{s_{H,i}, s_{G,i}\}\ \forall i$, it is not sufficient to obtain a solution. The set of $\{\mu, \{\nu_i\}, \{\lambda_l\}\}$ or $\{\mu, \{\nu_i\}, \{\sum_{j=i}^{K} \lambda_j\}\}$ needs to further satisfy other KKT conditions to solve (3). The reciprocal of $\sum_{j=i}^{K} \lambda_j$ is called the water level at epoch i for the entire process from epoch 1 to K. Thus, it is an important condition that the water level at epoch i depends on the duration of the process (e.g., the water level at epoch i is normally different for processes $[1, K1]$ and $[1, K2]$ where $K1$ and $K2$ are arbitrary ending epoch indexes). However, for system (3) in the original variables and the dual variables, to the best of the authors' knowledge, there is no existing method reported in the open literature to obtain an exact solution. The fact that the first equation in (3) is in the summation form $s_{H,i} + s_{G,i}$ introduces greater complexities to determine an optimal allocation solution from the harvested energy $\{s_{H,i}\}$ and from the smart grid power $\{s_{H,i}\}$, respectively.

III. Hybrid Power Allocation Algorithm 1 for Maximum Throughput

Since the proposed algorithms are based on GWF and GWFPP, they are concisely introduced as follows.

In [18], we presented a GWF approach for solving generalized radio resource allocation problems. As an extension, let L and K be two positive integers, and $L \le K$ denote the index of the starting channel and the ending channel, respectively. Then, $K - L + 1$ is the total number of channels. Let P denote the total power for allocation. GWF can be regarded as a mapping from the point of parameters$\{L, K, \{w_i\}_{i=L}^K, \{a_i\}_{i=L}^K, P\}$ to the solution $\{s_i\}_{i=L}^K$ and the important water-level step index k^*. That is, it can be written as a formal expression as follows [18]:

$$\left\{\{s_i\}_{i=L}^K, k^*\right\} = \text{GWF}\left(L, K, \{w_i\}_{i=L}^K, \{a_i\}_{i=L}^K, P\right). \tag{4}$$

Since we often only use the first part, i.e., $\{s_i\}_{i=L}^K$ from GWF, we also write

$$\{s_i\}_{i=L}^K = \text{GWF}\left(L, K, \{w_i\}_{i=L}^K, \{a_i\}_{i=L}^K, P\right)|_I. \tag{5}$$

Note that, for conciseness and without confusion from the context, we may write the right-hand side of the expression as GWF(L, K) to emphasize time stages from L to K.

Let $\overline{P}_i$ denote the peak power constraint for the ith channel, then GWFPP can be expressed as [18]

$$\begin{aligned}
\{s_i\}_{i=1}^K &= \text{GWFPP}\left(1, K, \{w_i, a_i, \overline{P}_i\}_{i=1}^K, P\right)|_I \\
E &= \text{GWFPP}\left(1, K, \{w_i, a_i, \overline{P}_i\}_{i=1}^K, P\right)|_{II}
\end{aligned} \tag{6}$$

where E is the final index set in which there is no peak power constraint [18]. Furthermore, for convenience, E may be written as $\{i_t | 1 < i_1 < \cdots < i_{|E|} \le K\}$, where $|E|$ is the cardinality of the set E. Thus

$$\left\{\{s_i\}_{i=1}^K, E\right\} = \text{GWFPP}\left(1, K, \{w_i, a_i, \overline{P}_i\}_{i=1}^K, P\right).$$

Without confusion, GWFPP$(1, K, \{w_i, a_i, \overline{P}_i\}_{i=1}^K, P)|_I$ can be regarded as GWFPP$(1, K, \{w_i, a_i, \overline{P}_i\}_{i=1}^K, P)$, due to a subordinate state of the final index set E.

In this paper, we propose a novel algorithm, i.e., HPA1, to solve problem (2) using the GWF approach. The pseudocode of the proposed HPA1(K) is stated in Algorithm 1 at the end of this paper. In the remainder of this section, algorithm description and optimality analysis will be presented.

Algorithm 1 Pseudocode for HPA1

1: Initialize:
  $L = 1, K, E_{(G,\text{total})}, \{w_k, a_k, \overline{P}_k = E_{G,k}, E_{\text{in}}(k)\}_{k=1}^{K}$;
2: Prepare:
  $\text{HPA1}(K)|_1 = \{\{s^*_{G,k}\}_{k=1}^{K}, E\} = \text{GWFPP}(\{a_i\}_{i=1}^{K})$;
3: Update:
  $\{a'_k \leftarrow (1/((1/a_k) + s^*_{G,k}))\}_{k=1}^{K}$;
4: Output the result for the epoch 1:
  $\text{RGWF}(L) = E_{\text{in}}(1)$;
5: **for** $L = 2:1:K$ **do**
6:   Input: $\{E_{\text{in}}(L), w_L, a'_L\}$;
7:   $\{s'_{H,k}\}_{k=1}^{L-1} = \text{RGWF}(L-1)$;
8:   **for** $n = L:-1:1$ **do**
9:     $W = \{w_j\}_{j=n}^{L}; A = \{a'_j\}_{j=n}^{L}$;
10:     $S_T = \sum_{j=n}^{L-1} s'_{H,j} + E_{\text{in}}(L)$;
11:     $\{\{s_{H,k^*}\}_{k=n}^{L}, k^*\} = \text{GWF}(n, L, W, A, S_T)$;
12:     $k^*_{\text{e}} = \max\{k | s'_{H,k} > 0, 1 \le k \le n-1\}$;
13:     **if** $(1/a'_{k^*} w_{k^*}) + (s_{H,k^*}/w_{k^*}) \ge (1/a'_{k^*_{\text{e}}} w_{k^*_{\text{e}}}) + (s_{H,k^*_{\text{e}}}{}'/w_{k^*_{\text{e}}})$ **then**
14:       output:
      $\text{RGWF}(L) = \{s'_{H,1}, \ldots, s'_{H,n-1}, s^*_{H,n}, \ldots, s^*_{H,L}\}$,
15:       Move to the next epoch, i.e., go to Line 18;
16:     **end if**
17:   **end for**
18: **end for**
19: **if** $E = \emptyset$ or $(1/a_{i_1} w_{i_1}) + ((s_{H,i_1} + s^*_{G,i_1})/w_{i_1}) \ge (1/a_K w_K) + ((s_{H,K} + s^*_{G,K})/w_K)$, where $s_{H,i}$ is the ith member of $\text{RGWF}(K)$ **then**
20:   output of $\text{HPA1}(K)$:
  $\{s^*_{G,k}\}_{k=1}^{K} = \text{HPA1}(K)|_I = \text{GWFPP}$ from Line 2;
  $\{s^*_{H,k}\}_{k=1}^{K} = \text{HPA1}(K)|_{II} = \text{RGWF}(K)$;
21: **end if**
22: $\text{HPA1}(K)|_{II} = \{\{s^*_{H,k}\}_{k=1}^{i_1-1}, \text{RGWF}(\{(1/(1/a_k w_k) + (([\text{HPA1}(K)]|_I)_k/w_k))\}_{k=i_1}^{K})$;
  $\text{HPA1}(K)|_I = \{\{s^*_{G,k}\}_{k=1}^{i_1-1}, \text{GWFPP}(\{(1/(1/a_k w_k) + (([\text{HPA1}(K)]|_{II})_k/w_k))\}_{k=i_1}^{K})$.

A. Two Parts of HPA1(K) and Their Noncommutativity

The proposed $\text{HPA1}(K)$ consists of two parts: $\text{HPA1}(K)|_I$ for smart grid power allocation, i.e.,

$$\{s^*_{G,k}\}_{k=1}^{K} = \text{HPA1}(K)|_I \tag{7}$$

and $\text{HPA1}(K)|_{II}$ for harvested energy power allocation portion, i.e.,

$$\{s^*_{H,k}\}_{k=1}^{K} = \text{HPA1}(K)|_{II}. \tag{8}$$

From the definition of $\text{HPA1}(K)$, it is seen that $\text{HPA1}(K)|_I$ is Algorithm GWFPP. GWFPP is used twice in the proposed $\text{HPA1}(K)$. The final index set E also appears twice. To distinguish between the two E's, the first E is also denoted by $E(1)$ and i_1 in $E(1)$ by $i_1(1)$. Similarly, we also have $E(2)$ and $i_1(2)$. In addition, $\text{HPA1}(K)|_{II}$ is Algorithm RGWF [19], in essence, with the only difference of the updated "step depths" or the updated channel gains. The implemented order for GWFPP and RGWF is using GWFPP as $\text{HPA1}(K)|_I$ to compute the initial distribution of the power from the power grid, then RGWF as $\text{HPA1}(K)|_{II}$ to compute the allocation of the power from energy harvesting, and finally using GWFPP again, under the condition, to adjust the distributed power from the power grid and then determine their allocation in the updated $\text{HPA1}(K)|_I$. In this way, the completed optimal solution to the proposed problem is obtained. Simply speaking, GWFPP is used twice, and between them, RGWF is used once. The following lemmas are proposed to study the optimality of HPA1.

Lemma 1: $\text{HPA1}(K)|_I$ can compute the optimal solution of problem (2) with finite loops, under $\sum_{i=1}^{K} E_{\text{in}}(i) = 0$.

Since $\text{HPA1}(K)|_I$ is Algorithm GWFPP, which has been discussed in detail in [18], the proof of Lemma 1 can be referred to [18, Prop. 3.1].

Lemma 2: $\text{HPA1}(K)|_{II}$ can compute the optimal solution of problem (2) with finite loops, under $E_{(G,\text{total})} = 0$.

Since $\text{HPA1}(K)|_{II}$ is Algorithm RGWF, which has been discussed in detail in [19], the proof of Lemma 2 can be referred to [19, Prop. 2].

It is seen that GWFPP and RGWF can be regarded as two functions in

$$\left\{1, K, \{w_i, a_i, E_{G,i}, E_{\text{in}}(i)\}_{i=1}^{K}, E_{G,\text{total}}\right\}$$

respectively. If

$$\left\{1, K, \{w_i, E_{G,i}, E_{\text{in}}(i)\}_{i=1}^{K}, E_{(G,\text{total})}\right\}$$

are kept unchanged except $\{a_i\}$, GWFPP can be written as $\text{GWFPP}(\{a_i\})$ to emphasize the relationship of the function in $\{a_i\}$; this is done as well with RGWF. Since these two functions are the set-valued functions, the kth evaluated value of the first function GWFPP is labeled as $[\text{GWFPP}(\{a_i\})]|_k$ or simply $[\text{GWFPP}]|_k$, as is that of the second function RGWF. We use HPA to denote the operation using GWFPP to allocate the grid power first, followed by RGWF to allocate the harvested energy. It can be expressed as

$$\begin{aligned}\text{HPA} &= (\text{RGWF} \circ \text{GWFPP}) \\ &\triangleq \text{RGWF}\left(\left\{\frac{1}{\frac{1}{a_i} + [\text{GWFPP}]\,|_i}\right\}\right).\end{aligned} \tag{9}$$

The second composite function reverses the order of HPA, denoted by HPA-R. It can be expressed as

$$\begin{aligned}\text{HPA-R} &= (\text{GWFPP} \circ \text{RGWF}) \\ &\triangleq \text{GWFPP}\left(\left\{\frac{1}{\frac{1}{a_i} + [\text{RGWF}]\,|_i}\right\}\right).\end{aligned} \tag{10}$$

Thus, using HPA-R can also output the power, respectively, from the energy harvesting and the power grid. However, the composite operation above does not satisfy the commutative law, i.e.,

$$\text{HPA} \neq \text{HPA-R}. \tag{11}$$

This point is accounted for by the following example.

Example III.1:

$$\begin{aligned} &\max_{\{(s_{H,i},s_{G,i})\}_{i=1}^{2}} && \sum_{i=1}^{2} \log(1 + (s_{H,i} + s_{G,i}) \\ &\text{subject to} && s_{H,i} \geq 0, \quad i = 1, 2 \\ &&& 0 \leq s_{G,1} \leq 2 \\ &&& 0 \leq s_{G,2} \leq 0.5 \\ &&& \sum_{i=1}^{l} s_{H,i} \leq l, \quad l = 1, 2 \\ &&& \sum_{i=1}^{2} s_{G,i} \leq 2.2. \end{aligned} \tag{12}$$

According to the definitions of HPA and HPA-R, the output of HPA is $\{s_{H,1} = 0.4, s_{H,2} = 1.6; s_{G,1} = 1.7, s_{G,2} = 0.5\}$, at which the objective function value is $\log 9.61$, whereas that of HPA-R is $\{s_{H,1} = 1, s_{H,2} = 1; s_{G,1} = 1.7, s_{G,2} = 0.5\}$, at which the objective function value is $\log 9.25$. Thus, the commutative law does not hold for this example. Moreover, HPA-R cannot guarantee to find the optimal solution. It further shows that the target problem cannot be decomposed into two decoupled subproblems in the two classes of power.

The pseudocode for the proposed HPA1 to solve the target problem (2) is listed in Algorithm 1 at the end of this paper. In Line 3, the step heights are updated, which are formed by the original fading gains and the power levels allocated in $\text{HPA1}(K)|_I$. From Lines 4 to 22, $\text{HPA1}(K)$ computes the optimal solution for the harvested energy part, i.e., $\text{HPA1}(K)|_{II}$ to complete the computation.

Note that, based on GWFPP, the term $(1/(a_i + s_{G,i})/w_i)$ denote the overall step depth after the power allocation of the power grid. Therefore, the reciprocal of $(1/(a_i + s_{G,i}))$ is $(a_i/(1 + a_i S_{G,i}))$, which is equivalent to the channel gain used by RGWF in [18].

The proposed algorithm eliminates the procedures to solve the nonlinear system (3) in multiple variables and dual variables, provides exact solutions via finite computation steps, and offers helpful insights to the problem and the solution.

B. Optimality of HPA1

Here, optimality of the proposed HPA1 is discussed.

Remark 1: $\text{HPA1}|_{II}$ is an optimal dynamic power distribution process. The dynamics of this recursive process are shown by the generalized state equation, i.e.,

$$\begin{aligned} &\text{HPA1}(L + 1)|_{II} \\ &\quad = [\text{HPA1}(L)|_{II}, \text{GWF}(n + 1, L + 1)|_I] \\ &\qquad\qquad \text{for } L = 1, \ldots, K - 1 \end{aligned} \tag{13}$$

where n is the index of the starting epoch of the currently processing window [19]. Note that the concept of dynamic processes is not identical to that of dynamic programming. The value of n is determined by $\text{HPA1}(L)|_{II}$. In this process, $\text{HPA1}(L)|_{II}$ can be regarded as the generalized system state at the time stage (or epoch) L, $\text{GWF}(n + 1, L + 1)$ can be regarded as the generalized system control at the time stage (or epoch) L, and then $\text{HPA1}(L + 1)|_{II}$, as a state at the next time stage, can be derived or determined from its previous state and control. Due to optimality of $\text{HPA1}(L)|_{II}$ from Lemma 2, for any L, the proposed algorithm is indeed an optimal and efficient forwarding dynamic recursive water-filling algorithm.

Since Lemma 2 guarantees optimality of $\text{HPA1}(K)|_{II}$ under the special condition, as does Lemma 1, we may obtain the following conclusion of $\text{HPA1}(K)$.

Proposition 1: HPA1 can compute the optimal exact solution to problem (2) within finite loops.

Proof of Proposition 1: First, $\text{HPA1}(K)|_I$ is implemented. Thus, it is equivalent to Lemma 1 being used. According to Lemma 1, for problem (2) under $\sum_{i=1}^{K} E_{\text{in}}(i) = 0$, there exist the optimal solution $\{s_{G,i}\}_{i=1}^{K}$ and the dual variables $\{\lambda^{(1)}, \{\nu_i^{(1)}, \mu_i^{(1)}\}_{i=1}^{K}\}$ such that they satisfy the following KKT conditions:

$$\begin{cases} \frac{1}{\frac{1}{a_i w_i} + \frac{s_{H,i} + s_{G,i}}{w_i}} = \lambda^{(1)} + \nu_i^{(1)} - \mu_i^{(1)} \quad \forall i \\ \mu_i^{(1)} s_{G,i} = 0, s_{G,i} \geq 0, \mu_i^{(1)} \geq 0 \quad \forall i \\ \nu_i^{(1)} (s_{G,i} - E_{G,i}) = 0, s_{G,i} \leq E_{G,i}, \nu_i^{(1)} \geq 0 \quad \forall i \\ \lambda^{(1)} \left(\sum_{i=1}^{K} s_{G,i} - E_{(G,\text{total})} \right) = 0 \\ \sum_{i=1}^{K} s_{G,i} \leq E_{(G,\text{total})}, \lambda^{(1)} \geq 0. \end{cases}$$

Second, $\text{HPA1(K)}|_{II}$ is implemented. Thus, it is equivalent to Lemma 2 being used with the updated "step depths" $(1/a_i w_i) + (s_{G,i}/w_i)$ or the updated channel gains $\{a_i/(1 + a_i s_{G,i})\}$.

According to Lemma 2 with the updated channel gains, for problem (2) under $E_{(G,\text{total})} = 0$, there exist the optimal solution $\{s_{H,i}\}_{i=1}^{K}$ and the dual variables $\{\lambda_i^{(2)}, \mu_i^{(2)}\}_{i=1}^{K}$ such that they satisfy the following KKT conditions:

$$\begin{cases} \frac{1}{\frac{1}{a_i w_i} + \frac{s_{H,i} + s_{G,i}}{w_i}} = \sum_{k=i}^{K} \lambda_k^{(2)} - \mu_i^{(2)} \quad \forall i \\ \mu_i^{(2)} s_{H,i} = 0, s_{H,i} \geq 0, \mu_i^{(2)} \geq 0 \quad \forall i \\ \lambda_i^{(2)} \left(\sum_{k=1}^{i} s_{G,k} - \sum_{k=1}^{i} E_{\text{in}}(k) \right) = 0 \\ \sum_{k=1}^{i} s_{G,k} \leq \sum_{k=1}^{i} E_{\text{in}}(k), \lambda_i^{(2)} \geq 0 \quad \forall i. \end{cases}$$

On one hand, if $E = \emptyset$ (the empty set), let $\lambda_G = 0$; $\nu_{G,i} = (1/(1/a_i w_i) + ((s_{H,i} + s_{G,i})/w_i)) \geq 0$ and $\mu_{G,i} = 0$, as $1 \leq i \leq K$. Moreover, let $\lambda_{H,i} = \lambda_i^{(2)}$, and $\mu_{H,i} = \mu_i^{(2)}$, for any i. Note that this set E aforementioned is obtained, when GWFPP is used at the first time.

On the other hand, if $E \neq \emptyset$ and $\exists i_1 \in E$ such that $(1/a_{i_1} w_{i_1}) + ((s_{H,i_1} + s_{G,i_1})/w_{i_1}) \geq (1/a_K w_K) + ((s_{H,K} + s_{G,K})/w_K)$, let $\lambda_G = (1/(1/a_{i_1} w_{i_1}) + ((s_{H,i_1} + s_{G,i_1})/w_{i_1})$, $\nu_{G,i} = (1/(1/a_i w_i) + ((s_{H,i} + s_{G,i})/w_i)) - \lambda_G \geq 0$, and $\mu_{G,i} = 0$, as $s_{G,i} > 0$ or the case of both $s_{G,i} = 0$ and $E_{G,i} = 0$, under $1 \leq i \leq K$. Note that this set E mentioned previously is obtained when GWFPP is still used at the first time. Further, if $E \neq \emptyset$, then $i_1 \in E$ such that $(1/a_{i_1}w_{i_1})+((s_{H,i_1}+s_{G,i_1})/w_{i_1}) \geq (1/a_k w_k)+((s_{H,k}+s_{G,k})/w_k)$, where $s_{G,k}>0, 1 \leq k \leq K$. Note that this set E above is obtained, when GWFPP is used second time. It has been emphasized that, to distinguish between the two E's, the first E is also denoted by $E(1)$ and i_1 in $E(1)$ by $i_1(1)$. Similarly, we also have $E(2)$ and $i_1(2)$. Let $\lambda_G = (1/(1/a_{i_1}w_{i_1}) + ((s_{H,i_1}+s_{G,i_1})/w_{i_1}))$, $\nu_{G,i} = (1/(1/a_i w_i)+(s_{H,i}+s_{G,i}/w_i)) - \lambda_G \geq 0$, and $\mu_{G,i} = 0$, as $s_{G,i}>0$ or the case of both $s_{G,i}=0$ and $E_{G,i}=0$, under $1 \leq i \leq K$. Furthermore, if $s_{G,i}=0$ and $E_{G,i}>0$, let $\nu_{G,i}=0, \mu_{G,i} = \lambda_G - (1/(1/a_i w_i) + ((s_{H,i}+s_{G,i})/w_i)) \geq 0$ with λ_G being defined earlier. Moreover, it has been noted that the water level for each of the epochs in the set $\{i | 1 \leq i < i_1(1)\}$ remains unchanged, although the adjustment is done or GWFPP is used twice; the difference between the water levels of epochs in the set $\{i | i_1(1) \leq i < i_1(2)\}$ is decreased; and the difference between the water levels of epochs in the set $\{i | i_1(2) \leq i \leq K\}$ is leaning forward to the same. Thus, $\{\lambda_{H,i}, \mu_{H,i}\}_{i=1}^K$ can be easily constructed, similar to those in [19] together with those distinguishing characteristics. That is, according to the two sets of KKT conditions, the definitions, the assigned/constructed values of the dual variables $\{\{\lambda_{H,i}, \mu_{H,i}, \nu_{G,i}, \mu_{G,i}\}, \lambda_G\}$, and the solutions $\{s_{H,i}, s_{G,i}\}_{i=1}^K$, it is seen that these dual variables and solutions also satisfy the following KKT conditions:

$$\begin{cases} \frac{1}{\frac{1}{a_i w_i}+\frac{s_{H,i}+s_{G,i}}{w_i}} = \lambda_G + \nu_{G,i} - \mu_{G,i} = \sum_{k=i}^K \lambda_{H,k} - \mu_{H,i} \\ \quad \forall i; \\ \mu_{H,i}s_{H,i} = 0, 0 \leq s_{H,i}, \mu_{H,i} \geq 0 \quad \forall i \\ \mu_{G,i}s_{G,i} = 0, 0 \leq s_{G,i}, \mu_{G,i} \geq 0 \quad \forall i \\ \nu_{G,i}(s_{G,i} - E_{G,i}) = 0, s_{G,i} \leq E_{G,i}, \nu_{G,i} \geq 0 \quad \forall i \\ \lambda_{H,l}\left(\sum_{i=1}^l s_{H,i} - \sum_{i=1}^l E_{\text{in}}(i)\right) = 0, \quad \lambda_{H,l} \geq 0 \\ \sum_{i=1}^l s_{H,i} \leq \sum_{i=1}^l E_{\text{in}}(i), \ 1 \leq l \leq K \\ \lambda_G\left(\sum_{i=1}^K s_{G,i} - E_{(G,\text{total})}\right) = 0, \quad \lambda_G \geq 0 \\ \sum_{i=1}^K s_{G,i} \leq E_{(G,\text{total})} \end{cases}$$

where this set of KKT conditions is of problem (2), the Lagrange function of which is

$$\begin{aligned} L&\left(\{s_{H,i}, s_{G,i}\}; \{\lambda_{H,i}, \mu_{H,i}\}, \{\lambda_G, \{\nu_{G,i}, \mu_{G,i}\}\}\right) \\ &= \sum_{i=1}^K w_i \log\left(1 + a_i(s_{H,i}+s_{G,i})\right) + \sum_{i=1}^K \mu_{H,i}s_{H,i} \\ &\quad - \sum_{l=1}^K \lambda_{H,l}\left(\sum_{k=1}^l s_{H,k} - \sum_{k=1}^l E_{\text{in}}(k)\right) \\ &\quad + \sum_{i=1}^K \mu_{G,i}s_{G,i} - \sum_{i=1}^K \nu_{G,i}(s_{G,i} - E_{G,i}) \\ &\quad - \lambda_G\left(\sum_{i=1}^K s_{G,i} - E_{(G,\text{total})}\right). \end{aligned}$$

In addition, we can observe that the general constraint qualification of problem (2) holds. Then, $\{s_{H,i}, s_{G,i}\}_{i=1}^K$ computed by the proposed HPA1 is the optimal solution to problem (2).

Therefore, Proposition 1 is proven.

In summary, first, HPA1 can compute the optimal solution only from the causal information in finite steps. It does not need to solve any nonlinear system, consisting of many equations and inequalities in multiple dual variables. Second, the relationship between HPA1$|_I$ and HPA1$|_{II}$ is determined, and the optimality of HPA1 stemming from HPA1$|_I$ and HPA1$|_{II}$ is revealed.

IV. Transmission Completion Time Minimization

Earlier, HPA1 was discussed as a recursive water-filling to efficiently solve the throughput maximization problem. Here, HPA1 is used to design two algorithms, i.e., HPA2 and HPA3, for efficiently solving the transmission completion time minimization problems.

Now, assume that the transmitter has B bits to be communicated to the receiver. Our objective is to minimize the time required to transmit these B bits.

This problem is categorized into two classes. The first class assumes that the completed time is taken at the end of the epochs as discrete-time points. Since T and $\{L_i\}$ are given, it needs to find the minimum index of the epochs for transmission. The second class assumes that the completed time is taken at a time point that is continuously located in the interval $[0, T]$ as a continuous straight segment.

A. *Discrete Transmission Completion Time Minimization*

The discrete transmission completion time minimization problem can be stated as follows: Assume N to be a positive integer and $N \leq K$, i.e.,

$$\begin{aligned} &\min_{\{\{s_{H,i}, s_{G,i}\}_{i=1}^N, N\}} \quad N \\ &\text{subject to} \quad \sum_{i=1}^N w_i \log(1 + a_i s_i) = B \\ &\qquad 0 \leq s_{H,i} \quad \forall i \\ &\qquad 0 \leq s_{G,i} \leq E_{G,i} \quad \forall i \\ &\qquad s_i = s_{H,i} + s_{G,i} \quad \forall i \\ &\qquad \sum_{i=1}^l s_{H,i} \leq \sum_{i=1}^l E_{\text{in}}(i) \quad \forall l \\ &\qquad \sum_{i=1}^K s_{G,i} \leq E_{(G,\text{total})}. \end{aligned} \tag{14}$$

We use the HPA1 machinery to design a recursive algorithm to solve (14), referred to as HPA2. The steps are described in the Algorithm 2 pseudocode at the end of this paper.

Algorithm 2 HPA2, Based on HPA1

1: $L = 1, K, B, P = E_{\text{in}}(1), E_G, E_{(G,\text{total})}, w_1$, and a_1;
2: Output the result for the epoch 1: $\{s'_{G,1}\} = \text{HPA1}(1)|_I = \text{GWFPP}(\{a_1\}),\ \{s'_{H,1}\} = \text{HPA1}(1)|_{II} = \text{RGWF}(\{a'_1 \leftarrow (a_1/(1+a_1 s'_{G,k}))\})$;
3: **if** $w_1 \log(1 + a_1(s'_{H,1} + s'_{G,1})) \geq B$ **then**
4: $\text{HPA2}(1)|_I = \{s^*_{G,1} = \{(s'_{G,1}/a_1(s'_{H,1} + s'_{G,1}))(2^{B/w_1} - 1)\}$;
5: $\text{HPA2}(1)|_{II} = \{s^*_{H,1} = \{(s'_{H,1}/a_1(s'_{H,1} + s'_{G,1}))(2^{B/w_1} - 1)\}$;
6: $N^* = 1$;
7: Exit the algorithm;
8: **end if**
9: **for** $L = 2 : 1 : K$ **do**
10: $\{E_{\text{in}}(L), w_L, a_L\}$;
11: $\{\{s'_{G,i}\}_{i=1}^L, E\} = \text{HPA1}(L)|_I$;
12: $\{s'_{H,i}\}_{i=1}^{L-1} = \text{HPA1}(L-1)|_{II}$;
13: **for** $n = L : -1 : 1$ **do**
14: $W = \{w_i\}_{i=n}^L; A = \{(a_i/(1 + a_i s'_{G,i}))\}_{i=n}^L$;
15: $S_T = \sum_{i=n}^{(L-1)} s'_{H,i} + E_{\text{in}}(L)$;
16: $\{\{s^*_{H,k}\}_{k=n}^L, k^*\} = \text{GWF}(n, L, W, A, S_T)$;
17: $(k^*_e, j^*_e) = \max\{k | s'_{H,k} > 0, 1 \leq k \leq n-1\}$;
18: **if** $(1/a_{k^*} w_{k^*}) + (s_{H,k^*}/w^*_k) \geq (1/a_{k^*_e} w_{(k^*_e)} + (s'_{H,k^*_e}/w_{k^*_e})$ **then**
19: **if** $E = \emptyset$ or $(1/a_{i_1} w_{i_1}) + ((s_{H,i_1} + s^*_{G,i_1})/w_{i_1}) \geq (1/a_L w_L) + ((s_{H,L} + s^*_{G,L})/w_L)$, where $s_{H,i}$ is the ith member of $\text{HPA1}(L)|_I$ **then**
20: $\text{HPA1}(L)|_{II} = \{s'_{H,k}\}_{k=1}^{(n-1)} \cup \{s^*_{H,k}\}_{k=n}^L$;
21: **end if**
22: Else, $\text{HPA1}(L)|_I = \{\{s^*_{H,k}\}_{k=1}^{i_1-1}, \text{RGWF}(\{1/(1/a_k w_k) + (([\text{HPA1}(L)]|_I)_k/w_k)\}_{k=i_1}^L); \text{HPA1}(L)|_I = \{\{s^*_{G,k}\}_{k=1}^{i_1-1}, \text{GWFPP}(\{1/(1/a_k w_k) + (([\text{HPA1}(L)]|_{II})_k/w_k\}_{k=i_1}^L)$;
23: $T_1 = \sum_{i=1}^{(n-1)} w_i \log(1 + a_i(s'_{H,i} + s'_{G,i}))$;
24: $T_2 = \sum_{i=n}^{L} w_i \log(1 + a_i(s^*_{H,i} + s'_{G,i}))$;
25: $T_3 = \sum_{i=n}^{L-1} w_i \log(1 + a_i(s^*_{H,i} + s'_{G,i}))$;
26: **if** $T_1 + T_2 \geq B$ **then**
27: $B_1 = B - T_1 - T_3$;
28: $\text{HPA2}(L)|_I = \{s'_{G,i}\}_{i=1}^{n-1} \cup \{s'_{G,j}\}_{j=n}^{L-1} \cup \{s^*_{G,L} = (s'_{G,L}/a_L(s^*_{H,L} + s'_{G,L}))(2^{B_1/w_L} - 1)\}$;
29: $\text{HPA2}(L)|_{II} = \{s'_{H,i}\}_{i=1}^{n-1} \cup \{s^*_{H,j}\}_{j=n}^{L-1} \cup \{s^*_{H,L} = (s^*_{H,L}/a_L(s^*_{H,L} + s'_{G,L}))(2^{B_1/w_L} - 1)\}$;
30: $N^* = L$;
31: Exit the algorithm.
32: **end if**
33: Move to the next epoch, i.e., go to Line 37;
34: **end if**
35: Move back to the previous epoch, i.e., go to Line 36;
36: **end for**
37: **end for**

Note that HPA2 consists of three output parts as $N^*, \text{HPA2}(N^*)|_I$ and $\text{HPA2}(N^*)|_{II}$. Compared with the steps of HPA1, in the first line of the algorithm, HPA2 introduces B as a parameter, whereas the others are the same. In line 9, HPA2 sequentially processes from the second epoch up to the Kth epoch to output the optimal value N^* and its optimal solution $\{\text{HPA2}(N^*), N^*\}$, which achieves the target rate of B bits. Similarly, the inner "For" loop updates power levels for the current processing epoch (L) and its previous $(L-n)$ epochs to form a processing window with the width of the window, i.e., $L - n + 1$. The GWF algorithm is applied to this window to find a common water level. Lines 23–25 define three temporary variables T_1, T_2, and T_3 to denote the transmitted bits from the beginning to epoch $n-1$, from epoch n to epoch L, from epoch n to epoch $L-1$, respectively, given as

$$
\begin{aligned}
T_1 &= \sum_{i=1}^{n-1} w_t \log\left(1 + a_i\left(s'_{H,i} + s'_{G,i}\right)\right) \\
T_2 &= \sum_{i=n}^{L} w_t \log\left(1 + a_i\left(s^*_{H,i} + s^*_{G,i}\right)\right) \\
T_3 &= \sum_{i=n}^{L-1} w_t \log\left(1 + a_i\left(s^*_{H,i} + s^*_{G,i}\right)\right). \qquad (15)
\end{aligned}
$$

Moreover, note that two new "If" clauses are inserted into the outer level "If" clause (for checking the water-level nondecreasing condition). In detail, the first new inner "If" clause is used to adjust the distributed power from the power grid. It has been used in Algorithm 1. The second new inner "If" clause is to check whether and how the transmitted bits reach B. Therefore, it is the normal exit of the algorithm (Line 31), where a typical solution is obtained. For convenience, the condition of this new "If" clause is called the criterion of HPA2. This emphasis is also due to the importance of the criterion in the following proposition.

To guarantee optimality of HPA2, the proposition is stated as follows.

Proposition 2: If no L exists such that the criterion in HPA2

$$T_1 + T_2 \geq B \qquad (16)$$

holds, then there is no solution to problem (14). If the criterion holds, then the obtained N^* is the optimal value, and the $\{\text{HPA2}(N^*), N^*\}$ is the exact optimal solution.

Proof of Proposition 2: For the given B, if no L exists such that the criterion in HPA2

$$T_1 + T_2 \geq B \qquad (17)$$

holds, it implies that the optimal value of problem (2) is strictly less than B, corresponding to Proposition 1. Thus, the first constraint of problem (14) never holds. As a result, there is no solution to problem (14).

Then, assume that there exist N^* and HPA1(N^*) such that

$$T_1 + \sum_{i=n}^{N^*} w_i \log\left(1 + a_i\left(s^*_{H,i} + s'_{G,i}\right)\right) \geq B$$

where

$$\left\{s'_{G,1}, \ldots, s'_{G,n-1}, s'_{G,n}, \ldots, s'_{G,N^*}\right\} = \text{HPA1}(N^*)|_I$$
$$\left\{s'_{H,1}, \ldots, s'_{H,n-1}, s^*_{H,n}, \ldots, s^*_{H,N^*}\right\} = \text{HPA1}(N^*)|_{II}.$$

According to the obtained N^* and the updated pair of $\{s^*_{H,N^*}, s^*_{G,N^*}\}$ from the HPA2 algorithm, the optimal value of the problem

$$\begin{aligned} \max_{\{s_{H,i}, s_{G,i}\}_{i=1}^N} \quad & \sum_{i=1}^{N} w_i \log\left(1 + a_i(s_{H,i} + s_{G,i})\right) \\ \text{subject to} \quad & 0 \leq s_{H,i} \quad \forall i \\ & 0 \leq s_{G,i} \leq E_{G,i} \quad \forall i \\ & \sum_{i=1}^{l} s_{H,i} \leq \sum_{i=1}^{l} E_{\text{in}}(i) \quad \forall l \\ & \sum_{i=1}^{N} s_{G,i} \leq E_{(G,\text{total})} \end{aligned} \tag{18}$$

is less than B, where HPA1(N) is the optimal solution to this problem, for $N = 1, \ldots, N^* - 1$. Hence, the optimal value of problem (14) is not less than N^*. Stemming from the statement, HPA2(N^*) is a feasible solution to problem (14), and further, N^* is the evaluated objective value of problem (14), by HPA2, at N^*. Thus, N^* is a feasible value, together with the fact that the optimal value of problem (14) is not less than N^*. As a result, N^* is the optimal value. Moreover, HPA2$(\cdot)$, computed or evaluated at the N^* epoch, is the exact optimal solution to problem (14). Therefore, Proposition 2 is proved.

Remark 2: HPA2 is an optimal dynamic recursive progressive process to compute the discrete transmission completion time minimization problem. This progressive process ends at the current epoch and then outputs the minimum completing time once the criterion is satisfied. Hence, it does not always need the solution/information of the entire process to the problem(s) or a tedious backlog of computation. Moreover, if no solution is output by HPA2, no solution exists, i.e., HPA2 is constructive. These advantages are obtained owing to the recursive feature of HPA1.

B. Continuous Transmission Completion Time Minimization

The continuous transmission completion time minimization problem can be stated as follows: Assume that t is a real number that N is an index variable of the epochs, and that $\mathbb{Z}$ is the set of integers; then, the corresponding objective function is written as

$$\begin{aligned} \min_{\{N, \{s_i\}_{i=1}^N, t\}} \quad & t \\ \text{subject to} \quad & 1 \leq N \leq K \\ & N \in \mathbb{Z} \\ & N_1(t) = \max\left\{N \middle| \sum_{i=1}^{N} L_i \leq t\right\} \\ & \sum_{l=1}^{N_1-1} w_l \cdot \log\left(1 + a_l \cdot (s_{H,l} + s_{G,l})\right) \\ & \quad + \left(\frac{t}{2} - \sum_{k=1}^{N_1-1} w_k\right) \\ & \quad \log\left(1 + a_{N_1}\left(s_{H,N_1} + s_{G,N_1}\right)\right) = B \\ & 0 \leq s_{H,i}, \quad \text{for } i = 1, \ldots, N \\ & 0 \leq s_{G,i} \leq E_{G,i} \quad \forall i \\ & 0 \leq t \leq T \\ & \sum_{i=1}^{l} s_{H,i} \leq \sum_{i=1}^{l} E_{\text{in}}(i) \quad \forall l \\ & \sum_{i=1}^{K} s_{G,i} \leq E_{(G,\text{total})}. \end{aligned} \tag{19}$$

If Lebesgue–Stieltjes integration [24] were used for problem (19), it would have made the expression more concise. However, to avoid introducing more abstract mathematical tools, the presented method is used. We use the proposed HPA2 to design another algorithm to solve the continuous transmission completion time minimization problem (19), referred to as HPA3. It has similar steps to those of HPA2, except some modification to Lines 28–30 or 4–6 described in the pseudocode of Algorithm 2, where Lines 4–6 are considered, as a trivial case, due to completing the task possibly at epoch 1. Lines 28–29 are combined to

$$\Delta t^* = \frac{2B_1}{\log\left(1 + a_L\left(s^*_{H,L} + s^*_{G,L}\right)\right)}. \tag{20}$$

Line 30 is changed to

$$\begin{aligned} & N^* = L \\ & t^* = \Delta t^* + \sum_{k=1}^{N^*-1} L_k = \Delta t^* + 2\sum_{k=1}^{N^*-1} w_k. \end{aligned} \tag{21}$$

The optimality proof of HPA3 can be obtained similarly to that of HPA2 (see Proposition 2). Therefore, its proof is ignored in this paper.

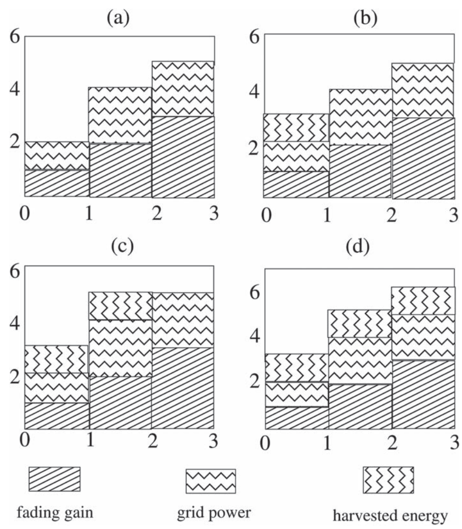

Fig. 2. Procedures to solve Example 1. (a) $s_{G,1} = 1$, $s_{G,2} = 2$, and $s_{G,3} = 2$. (b) RGWF(1) $= \{s_{H,1} = 1\}$. (c) RGWF(2) $= \{s_{H,1} = s_{H,2} = 1\}$. (d) RGWF(3) $= \{s_{H,1} = s_{H,2} = s_{H,3} = 1\}$ and HPA1(3).

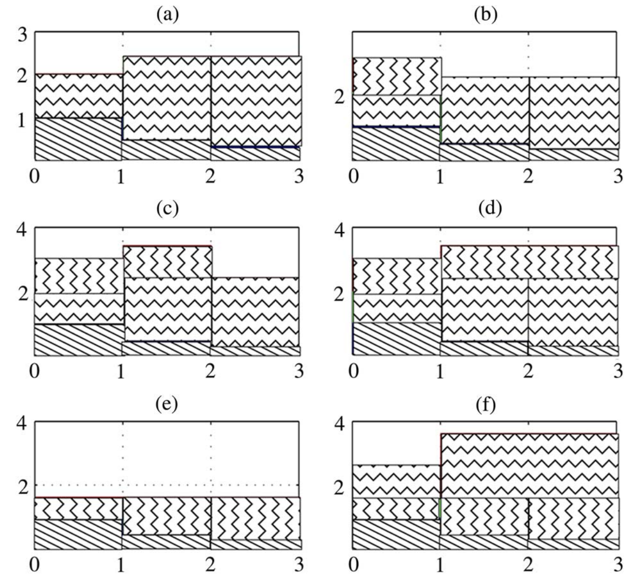

Fig. 3. Procedures to solve Example 2, where (a)–(d) corresponds to HPA1, whereas (e)–(f) corresponds to HPA-R. (a) $s_{G,1} = 1$, $s_{G,2} = (23/12)$, and $s_{G,3} = (25/12)$. (b) RGWF(1) $= \{s_{H,1} = 1\}$. (c) RGWF(2) $= \{s_{H,1} = s_{H,2} = 1\}$. (d) RGWF(3) $= \{s_{H,1} = s_{H,2} = s_{H,3} = 1\}$ and HPA1(3). (e) RGWF(3) $= \{s_{H,1} = (11/18), s_{H,2} = (20/18), s_{H,3} = (23/18)\}$. (f) GWFPP(3) $= \{s_{G,1} = 1, s_{G,2} = s_{G,3} = 2\}$ and HPA-R(3).

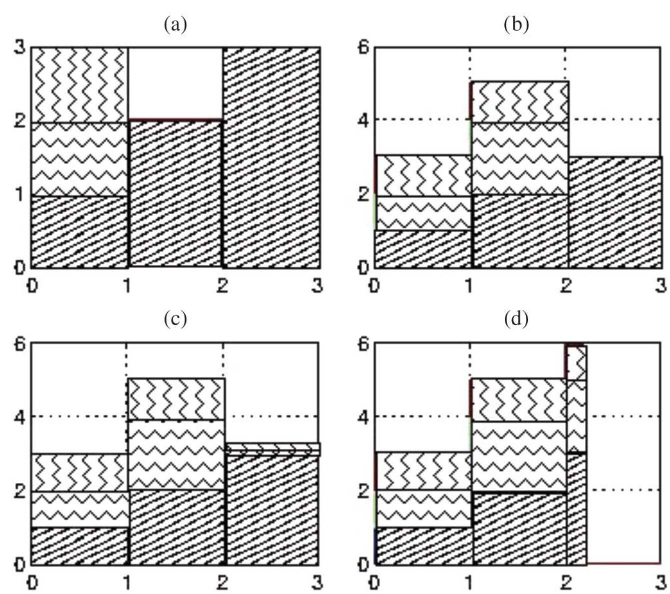

Fig. 4. Procedures to solve Example 3 by HPA2. (a) $s^*_{G,1} = 1$, and $s^*_{H,1} = 1$. (b) $s^*_{G,2} = 2$, and $s^*_{H,2} = 1$. (c) $sz^*_{G,3} = (2/15)$, and $s^*_{H,3} = (1/15)$. (d) $s^*_{G,1} = 1$, $s^*_{G,2} = s^*_{G,3} = 2$, $s^*_{H,i} = 1\ \forall i$, and the optimal value $t^* \doteq 2.1862$.

V. Numerical Examples and Computational Complexity

HPA1 does not need to wait for the full information to be available, but it can compute the exact optimal solution through finite computation for every subprocess that starts from epoch 1 and ends at epoch i, $i = 1, \ldots, K$. This point can also lean towards designing other efficient algorithms, such as algorithms HPA2 and HPA3, to compute the minimum transmission completion time. For the minimum transmission completion time problems, although they are nonconvex and nonsmooth optimization problems, our approach can effectively compute their exact global optimal solutions, without the use of any huge backlog of computation and nonsmooth analysis. At the same time, since HPA2 and HPA3 stem from HPA1 and they are nonconvex and mixing continuous and integer optimization variables, the computational complexity of HPA1 is analyzed and compared with that of primal–dual interior point method (PD-IPM), which has been regarded as an efficient optimization algorithm with great promise (see [25] and references therein).

A. Numerical Examples

Here, we first present three numerical examples to illustrate the procedures of the proposed algorithms (HPA1, HPA2, and HPA3), followed by the achieved throughput comparison with more general settings. For simple illustration of the first three examples, we assume that there are three epochs, each with unit length ($L_i = 1, i = 1, 2, 3$), i.e., the same weight ($w_i = 1/2, i = 1, 2, 3$). The logarithm operation has base 2 by default. For Figs. 2–4, the height of the slash-filled shadowed stairs denote the fading gains. The grid power and the allocated harvested power are illustrated by the height of the horizontal and vertical wave-filled shadows, respectively. The x-axis denotes the number of epochs, and the y-axis denotes the allocated power levels.

Example 1: Suppose the fading profile for the three epochs is $a_1 = 1$, $a_2 = 1/2$, and $a_3 = 1/3$. At the beginning of each epoch, unit energy is harvested ($E_{\text{in}}(i) = 1, i = 1, 2, 3$). Moreover, the upper bound constraint of power from the power grid is $E_{G,i} = i, i = 1, 2, 3$, and the entire power sum from the power grid is $E_{(G,\text{total})} = 5$.

First, implement HPA1$(3)|_I$, which gives HPA1$(3)|_I = \{s_{G,1} = 1, s_{G,2} = 2, s_{G,3} = 2\}$ as shown in Fig. 2(a). Second, epoch 1 is first scanned to output RGWF(1) $= s_{H,1} = 1$, as shown in Fig. 2(b). Now, we move to epoch 2 and output RGWF(2) $= s_{H,2} = 1$, as shown in Fig. 2(c). Similarly, for epoch 3, by applying RGWF(3), we have $s_{H,3} = 1$. Therefore, the algorithm HPA1(3) outputs the completed solution, as shown in Fig. 2(d).

Example 1 is calculated out without power-level adjustment by RGWF since the water-level nondecreasing condition is satisfied for all the epochs. The updated channel gains (fading channel gains) for the three epochs are continuously deteriorating. Therefore, the harvested energy at the beginning of each epoch is fully consumed in the current epoch.

Example 2: Suppose the fading profile for the three epochs is $a_1 = 1$, $a_2 = 2$, and $a_3 = 3$. The remaining assumptions are the same as in Example 1.

The solving procedures are shown in Fig. 3. First, implement HPA1$(3)|_I$, i.e., GWFPP. It is obtained that HPA1 $(3)|_I = \{s_{G,1} = 1, s_{G,2} = 23/12, s_{G,3} = 25/12\}$, as shown in Fig. 3(a). Second, epoch 1 is first scanned to output RGWF$(1) = s_{H,1} = 1$ as shown in Fig. 3(b). Now, we move to epoch 2 and output RGWF$(2) = s_{H,2} = 1$, as shown in Fig. 3(c). Similarly, for epoch 3, by applying RGWF(3), we have $s_{H,3} = 1$. Therefore, the algorithm HPA1(3) outputs the completed solution, as shown in Fig. 3(d). The maximum throughput is $\log(41^2/8) \doteq 7.72$.

If we apply HPA-R, i.e., the harvested energy is allocated first, we would meet with the power-level adjustment procedure during the use of RGWF(2) and RGWF(3) [19]. Therefore, the harvested energy at the beginning of each epoch attempts to flow to the later epochs because of the continuing improved channel fading condition, leading to the uniform water level of these three epochs, as shown in Fig. 3(e): $\{s_{H,1} = (11/18), s_{H,2} = (20/18), s_{H,3} = (23/18)\}$. Then, the grid power is allocated as $\{s_{G,1} = 1, s_{G,2} = s_{G,3} = 2\}$, as shown in Fig. 3(f). The corresponding objective value is $\log((47 \times 65^2)/(3^3 \times 6^2)) \doteq 7.67$. Therefore, the solution by HPA-R is not an optimal solution. The illustration also accounts for the maximum absolute difference of the water levels over all the epochs obtained by HPA-R being greater than that obtained by HPA1.

Example 3: All other assumptions are the same as those in Example 1. In addition, the information required for transmission is $B = 3$ bits (strictly speaking, B bits/Hz).

Epoch 1 is first scanned to output HPA1(1)$=\{s_{G,1}=1, s_{H,1}=1\}$, as shown in Fig. 4(a). Since $\log(1+2) < B(=3)$, we move to epoch 2, and apply HPA1(2) and HPA1(2)$=$ $\{s_{G,1}=1, s_{H,1}=1\}$ and $\{s_{G,2}=2, s_{H,2}=1\}$, as shown in Fig. 4(b). Since $\log(1+2) + \log(1+(1/2)\times 3) < B(=3)$, we continually move to epoch 3 and apply HPA1(3) and HPA1(3)$=\{s_{G,1}=1, s_{H,1}=1; s_{G,2}=2, s_{H,2}=1\}$ and $\{s_{G,3}= 2, s_{H,3}=1\}$. Since $\log(1+2)+\log(1+(1/2)\times 3) + \log(1+(1/3)\times 3) > B(=3)$, $B_1 = \log(16/15)$. Therefore, The optimal solution is $\{\{s^*_{H,1} = s^*_{H,2} = 1, s^*_{H,3} = (1/15)\}, \{s^*_{G,1} = 1, s^*_{G,2} = 2, s^*_{G,3} = (2/15)\}, N^* = 3\}$, and the optimal value is $N^* = 3$ by HPA2, as shown in Fig. 4(c). In addition, the minimizing continuous transmission completion time is $t^* = \Delta t^* + \sum_{k=1}^{N^*-1} L_k = 2\log(16/15) + 2$, and its solution is $\{\{s^*_{G,1}=1, s^*_{G,2} = s^*_{G,3} = 2\}, \{s^*_{H,1} = s^*_{H,2} = s^*_{H,3} = 1\}, t^*\}$ by HPA3, as shown in Fig. 4(d).

Example 4: PD-IPM is chosen for the purpose of comparison due to its competitiveness in computing the solutions to the convex optimization problems. It has been known that the proposed minimum transmission completion time problems are nonconvex and mixing continuous with integer variable optimization problems. To the best of the authors' knowledge, there is no algorithm reported in the open literature that can compute the exact solutions to these problems. As a result, we focus on the comparison of HPA1 and HPA-R with PD-IPM.

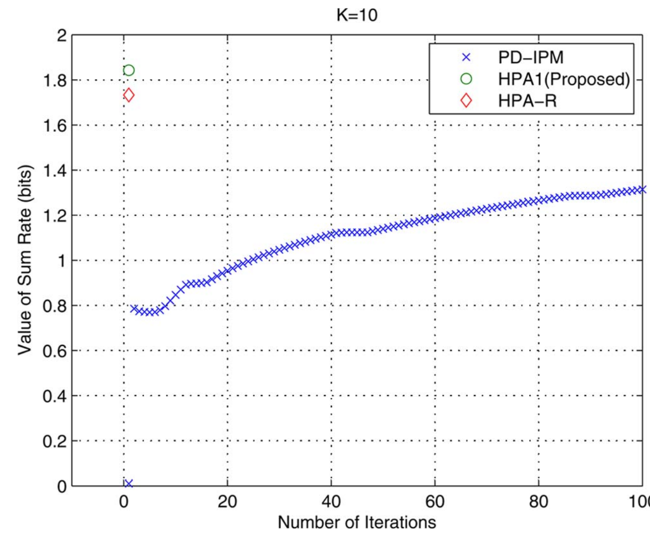

Fig. 5. Weighted sum rates (Unit: bits) of HPA1, HPA-R, and PD-IPM, as $K = 10$.

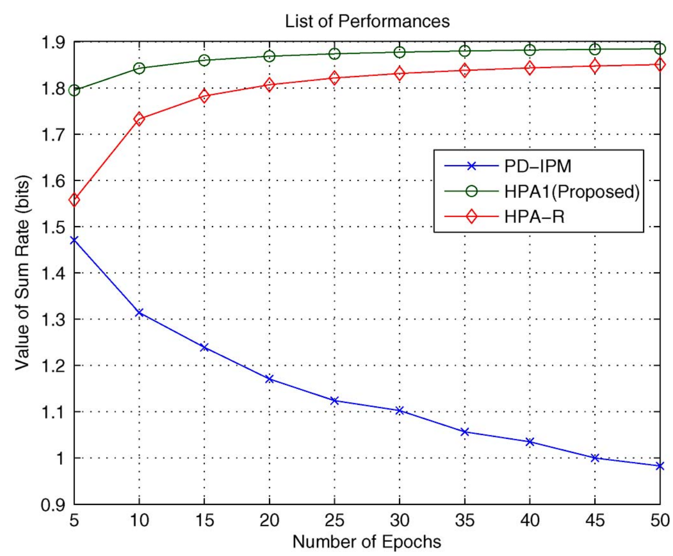

Fig. 6. Weighted sum rates (Unit: bits) of HPA1, HPA-R, and PD-IPM, as a function of K.

Figs. 5 and 6 compare the achieved throughput of PD-IPM, HPA-R, and HPA1 for the maximum throughput problem. In Fig. 5, /the number of epochs is selected as 10 $(K = 10)$. The throughput is shown with the iteration growth. For HPA-R and HPA1, they belong to the recursive algorithms. The number of iteration is simply marked as one to complete the calculation. The achieved throughput by using HPA1 and HPA-R is significantly higher than that of PD-IPM.

In Fig. 6, the achieved throughput is plotted as a function of the number of epochs. For PD-IPM, the number of iteration is fixed at 100. Channel gains are generated randomly using random variables with the standard Gaussian distribution. For convenience, $\{E_{\rm in}(k) = 6\ \forall k\}$. The sum power constraint of the power grid $E_{(G,\rm total)} = K$ and the peak power constraints $\{E_{G,k} = k\,\forall k\}$. A group of different weights are also generated randomly. The chosen parameters are assigned to all the three algorithms with the identical values for fair comparison. For the proposed HPA1 and HPA-R, the throughput increases with the increasing of the number of epochs and is much higher than the result from PD-IPM. In the simulated range, HPA1 always achieves higher throughput than HPA-R, particularly when the number of epochs is low. On the other side, with the increasing of the number of epochs, PD-IPM has more and more variables to be solved. Thus, it needs more iterations to achieve its converged sum rate value. This point leads to the fact that, when the number of iteration is fixed as 100, the achieved throughput is decreasing monotonically in the number of epochs.

B. Computational Complexity Analysis

HPA1 utilizes GWF $\sum_{L=1}^{K}(1+L)L/2$ times for RGWF(K); therefore, it needs $\sum_{L=1}^{K}\sum_{k=1}^{L}(8k+3) = K(K+1)(8K+25)/6$, i.e., $O(K^3)$ fundamental operations for utilizing GWF (see [18]). Since HPA1 also uses GWFPP twice, this usage needs $8K^2 + 14K$ fundamental operations (see [18]). Therefore, HPA1 needs $K(K+1)(8K+25)/6 + 8K^2 + 14K$, i.e., $O(K^3)$ fundamental operations. Therefore, the complexity of HPA1 is rather low $O(K^3)$. As a comparison, PD-IPM computes an ϵ solution, which is not the optimal solution. It still needs a polynomial computational complexity: $O(K^{3.5})\log(1/\epsilon)$ (see [25] and [26]). Hence, PD-IPM cannot guarantee to output the optimal solution by finite computation. Our method eliminates any linear search but outputs the exact optimal solution with finite computation.

Simply speaking, HPA1 needs a total of $O(K^3)$ basic operations to compute the optimal exact solution, whereas PD-IPM needs a total of $O(K^{3.5})\log(1/\epsilon)$ basic operations to compute an ϵ solution.

VI. CONCLUSION

For the general model of the optimal power allocation for wireless communications with the energy harvesting and the smart power grid coexisting systems, we proposed recursive algorithms to solve the radio resource allocation problems with more general and more complicated constraints than the existing problems. As a starting point, we reviewed the proposed GWF to solve the optimal power allocation problem with a sum power constraint. Then, GWF is used as a functional block to solve the problems with energy harvesting and grid power in fading channels for the objective function to maximize the data rate (HPA1) and to minimize the discrete transmission completion time (HPA2) and the continuous transmission completion time (HPA3), respectively.

Applying the developed RGWF from GWF recursively and comparing the water level at the current processing window with that of the previous epoch, HPA1, which includes the developed GWFPP, outputs the solution epoch by epoch to find the optimal power allocation policy. HPA2 and HPA3 are constructed based on HPA1 with additional comparison: to see whether the required information transmission amount is achieved. Significantly, the proposed algorithms own lower computational complexity. For example, HPA1 is of cubic polynomial computational complexity to compute an exact optimal solution. The number of iterations of all these proposed algorithms is finite. We have also obtained and strictly proved optimality of the proposed algorithms. Numerical examples are provided to illustrate the steps to obtain the exact optimal solutions via finite computation using the proposed algorithms.

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