Signals and Systems I

Lectures 10 &11

Last Lecture

- Trigonometric Fourier Series & Exponential Fourier Series
- LTI Systems, Eigen Function of LTI System (e^{st})
- LTI Systems & Fourier Series
- Some Properties of Fourier Series
- More Examples

Today

- Fourier Transform
- Fourier Transform Properties

Fourier Series vs Fourier Transform

Fourier series: Periodic signals	Fourier transform: More signals (including periodic ones)
$x_p = \sum D_n e^{j\omega_0 nt}$	$x(t) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$
$D_n = \frac{1}{T_0} \int_{<T_0>} x_p(t) e^{-j\omega_0 nt} dt$	$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$
$\lvert D_n \rvert \& \angle D_n$ for each n at $\underbrace{\omega_0 n}_{\text{freq.}}$	$\lvert X(j\omega) \rvert \& \angle X(j\omega)$ for all ω

Similar to Fourier Series, Fourier Transform has periodic spirals in form of $e^{j\omega t}$ and they are being amplified with $|X(j\omega)|$ and rotated with $\angle(X(j\omega)$

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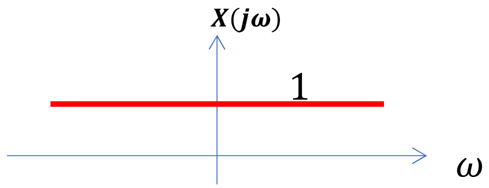
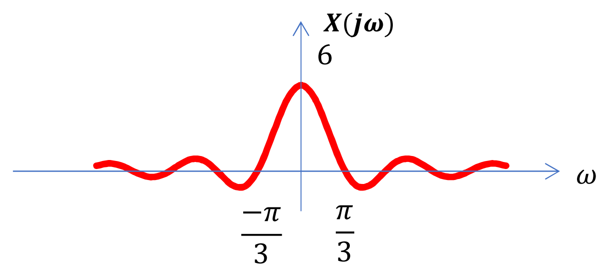
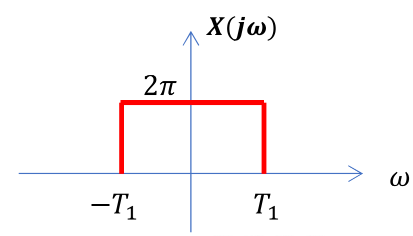
Similar to Fourier Series, Fourier Transform has periodic spirals in form of $e^{j\omega t}$ and they are being amplified with $|X(j\omega)|$ and rotated with $\angle(X(j\omega)$

What is the difference between the following FTs:

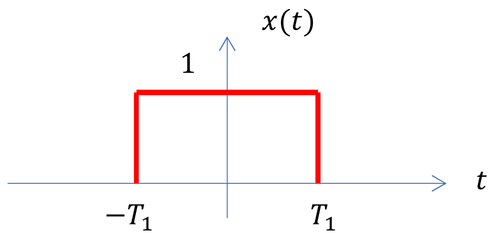






Fourier Transform

Example: Find and plot Fourier transform of the following signal:



$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Fourier Transform

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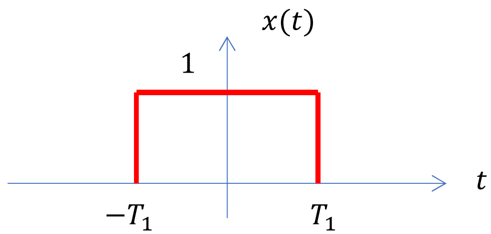


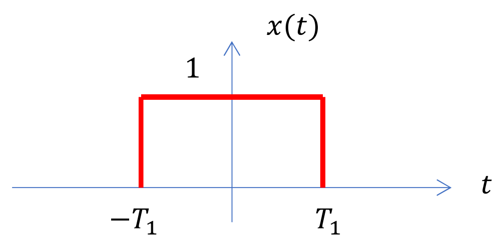
Answer:

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\ &= \int_{-T_1}^{T_1} 1 \times e^{-j\omega t}dt \\ &= \frac{e^{j\omega T_1} - e^{-j\omega T_1}}{j\omega} \\ &= \frac{2sin(\omega T_1)}{\omega} \rightarrow Sinc\ Structure \end{aligned}$$

Pulse in time is always a sinc in frequnecy

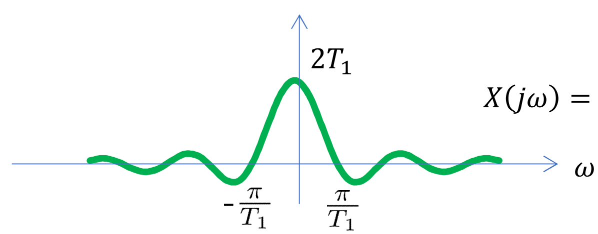
Fourier Transform

Find and plot Fourier transform for the following signal:



Answer:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$
$$= \int_{-T_1}^{T_1} 1.e^{-j\omega t}dt$$
$$= \frac{e^{j\omega T_1} - e^{-j\omega T_1}}{j\omega}$$
$$= \frac{2sin(\omega T_1)}{\omega} = \frac{2T_1 sin(\omega T_1)}{\omega T_1}$$



Pulse in time is always a sinc in frequnecy

Fourier Transform

Example: Find and plot Fourier transform of the following signal:

$$x(t) = e^{-at}u(t), \quad a > 0$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Fourier Transform

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$$x(t) = e^{-at}u(t), \quad a > 0$$

$$\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\
&= \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt \\
&= \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt \\
&= \frac{e^{-t(a+j\omega)}}{-(a+j\omega)}\bigg|_0^{\infty} \\
&= \frac{e^{-\infty(a+j\omega)}}{-(a+j\omega)} - \frac{1}{-(a+j\omega)} \quad \text{since } a>0,\ \lim_{t\to\infty} e^{-at} = 0 \text{ and } e^{j\omega t} \text{ doesn't have a limit, it rotates on a unit circle,} \\
&\qquad |e^{j\omega t}| = 1 \text{ for all } t \text{ even as } t \to -\infty \\
&= 0 - \frac{1}{-(a+j\omega)} \\
&= \frac{1}{a+j\omega}
\end{aligned}$$

Fourier Transform

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$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt \\ &= \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt \\ &= \frac{e^{-t(a+j\omega)}}{-(a+j\omega)}\bigg|_{0}^{\infty} \\ &= 0 - \frac{1}{-(a+j\omega)} \\ &= \frac{1}{a+j\omega} \end{aligned}$$

$$|X(j\omega)| = \frac{1}{\sqrt{\omega^2 + a^2}} \longrightarrow |X(j\omega)| = |X(-j\omega)|$$

$$\angle X(j\omega) = \angle \frac{1}{j\omega + a} = -\angle(j\omega + a) = -tan^{-1}(\frac{\omega}{a})$$

Fourier Transform

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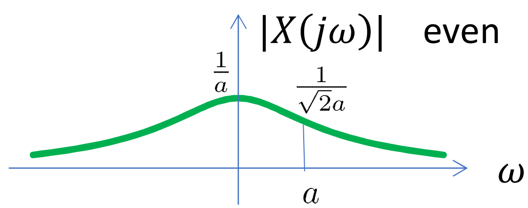
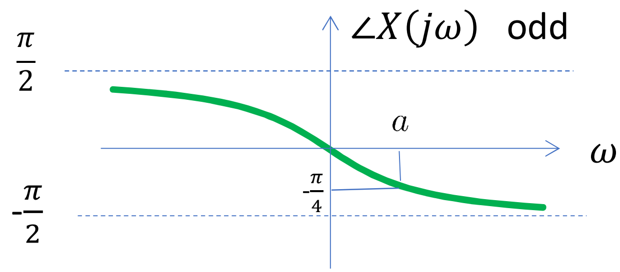




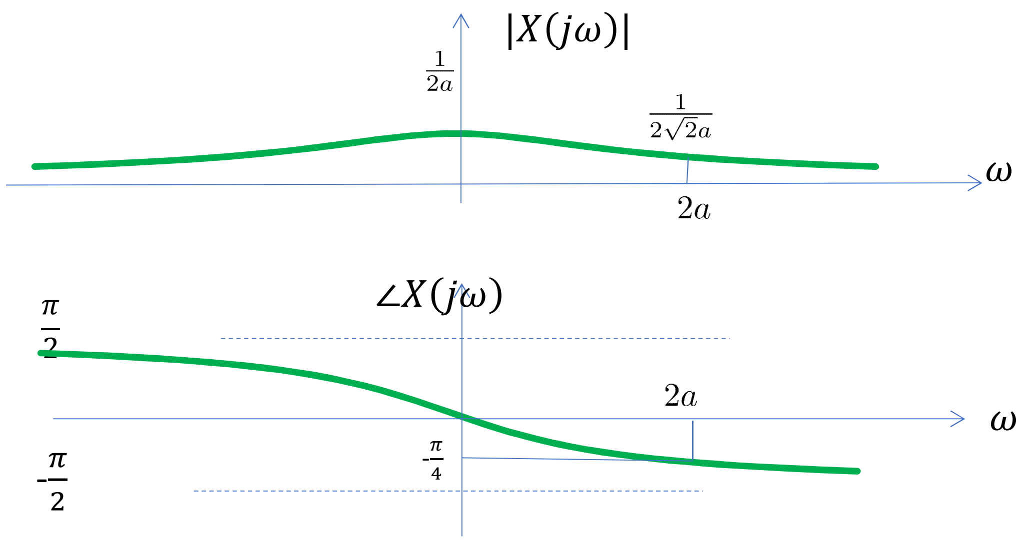
For a real signal in time, the absolute value of FT is even and the phase is odd.

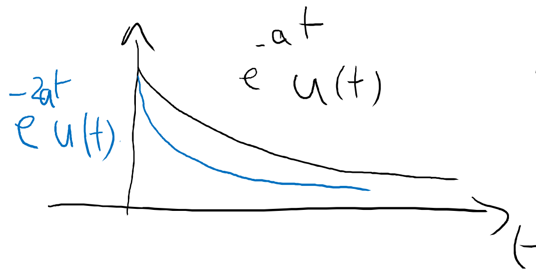
How will this FT change as the value of a grows?

Fourier Transform

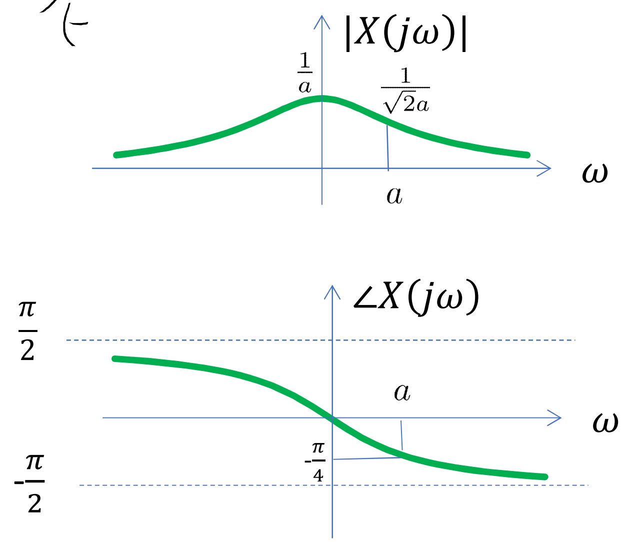
Example: Find and plot Fourier transform of the following signal:

$$x(t) = e^{-2at}u(t), \quad a > 0$$





$$x(t) = e^{-at}u(t), \quad a > 0$$



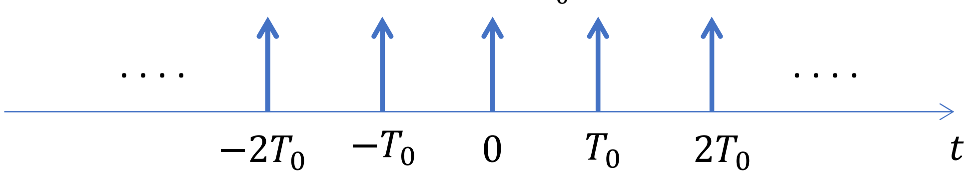
Connection between Fourier Series (in limit) and Fourier Transform

Remember the FS of impulse train:

$$\delta_{T_0}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0), \qquad \omega_0 = \frac{2\pi}{T_0}$$

$$D_n = \frac{1}{T_0}\int_{T_0} \delta_{T_0}(t) e^{-jn\omega_0 t} dt = \int_{\frac{-T}{2}}^{\frac{-T}{2}} \delta(t) e^{-jn\omega_0 t} = \frac{1}{T_0}$$

$$x(t) = \delta_{T_0}(t)$$



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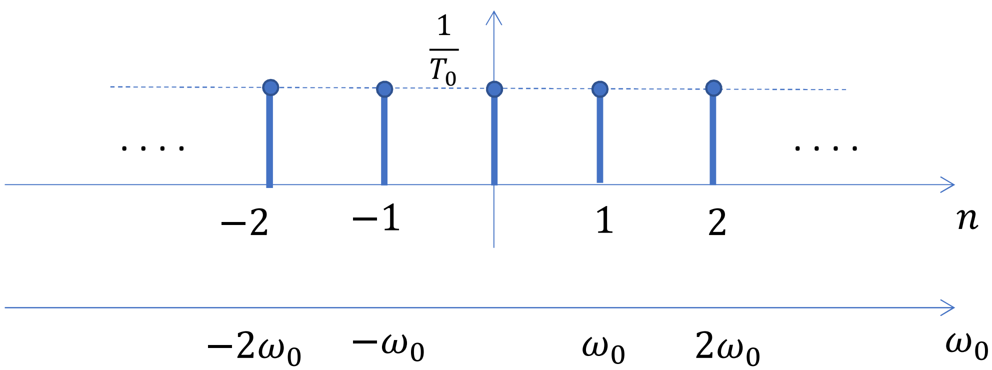

What will happen as T_0 grows to infinity?

Connection between Fourier Series (in limit) and Fourier Transform

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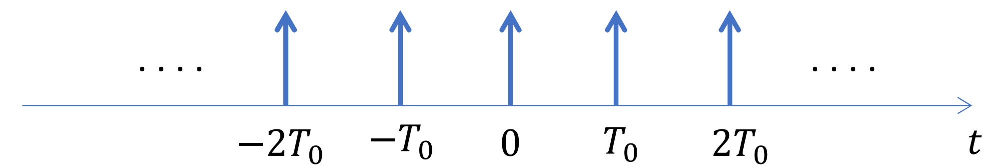

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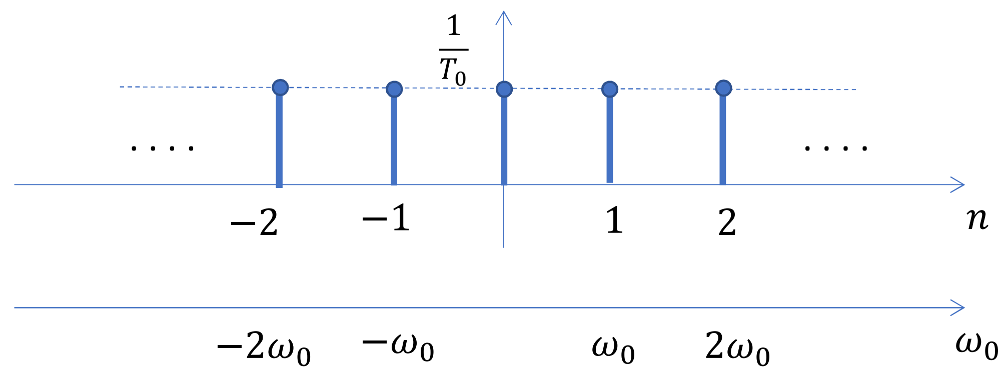
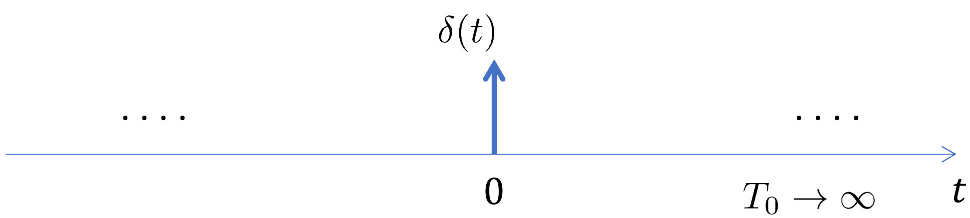


What will happen as T_0 grows to infinity? $\delta(t) = \lim_{T_0 \to \infty} \delta_{T_0}(t)$



What is FT of $\delta(t)$?

$$\Delta(j\omega) = \int \delta(t)e^{-j\omega t}dt$$

Connection between Fourier Series (in limit) and Fourier Transform

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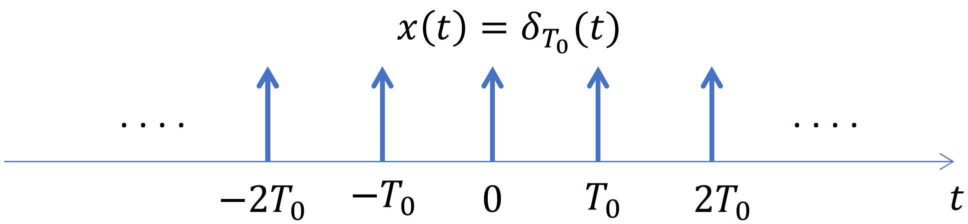
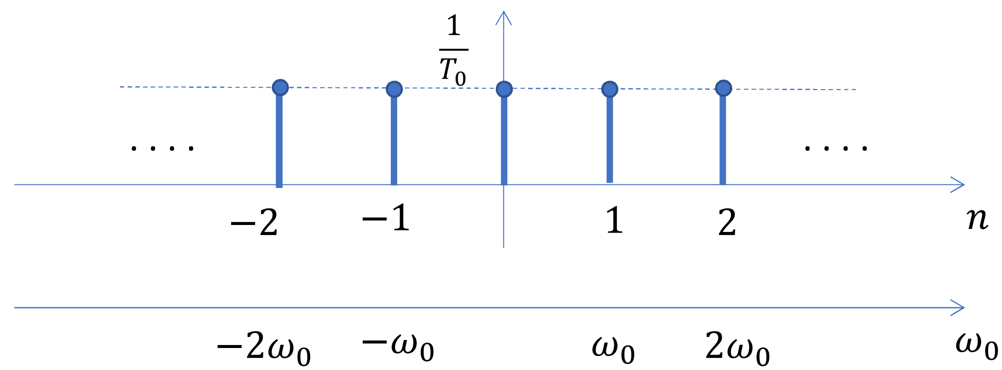
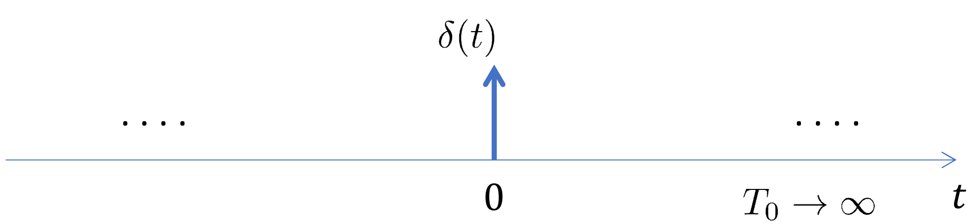


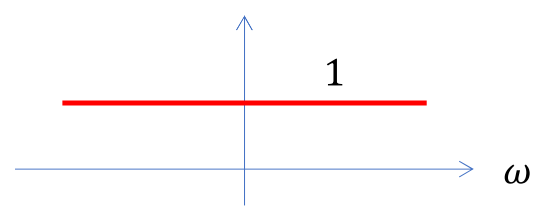


What will happen as T_0 grows to infinity?

$$\delta(t) = \lim_{T_0\to\infty} \delta_{T_0}(t)$$



$$\Delta(j\omega) = \int \delta(t) e^{-j\omega t} dt = 1$$

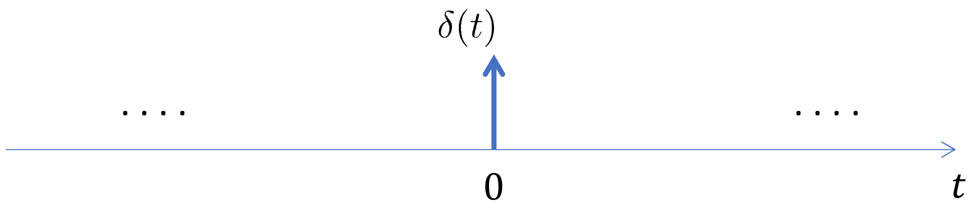


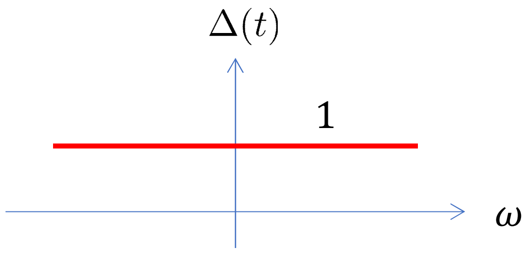
$\Delta(j\omega) = \lim_{T_0\to\infty} T_0 D_n$

Fourier Transform of delta

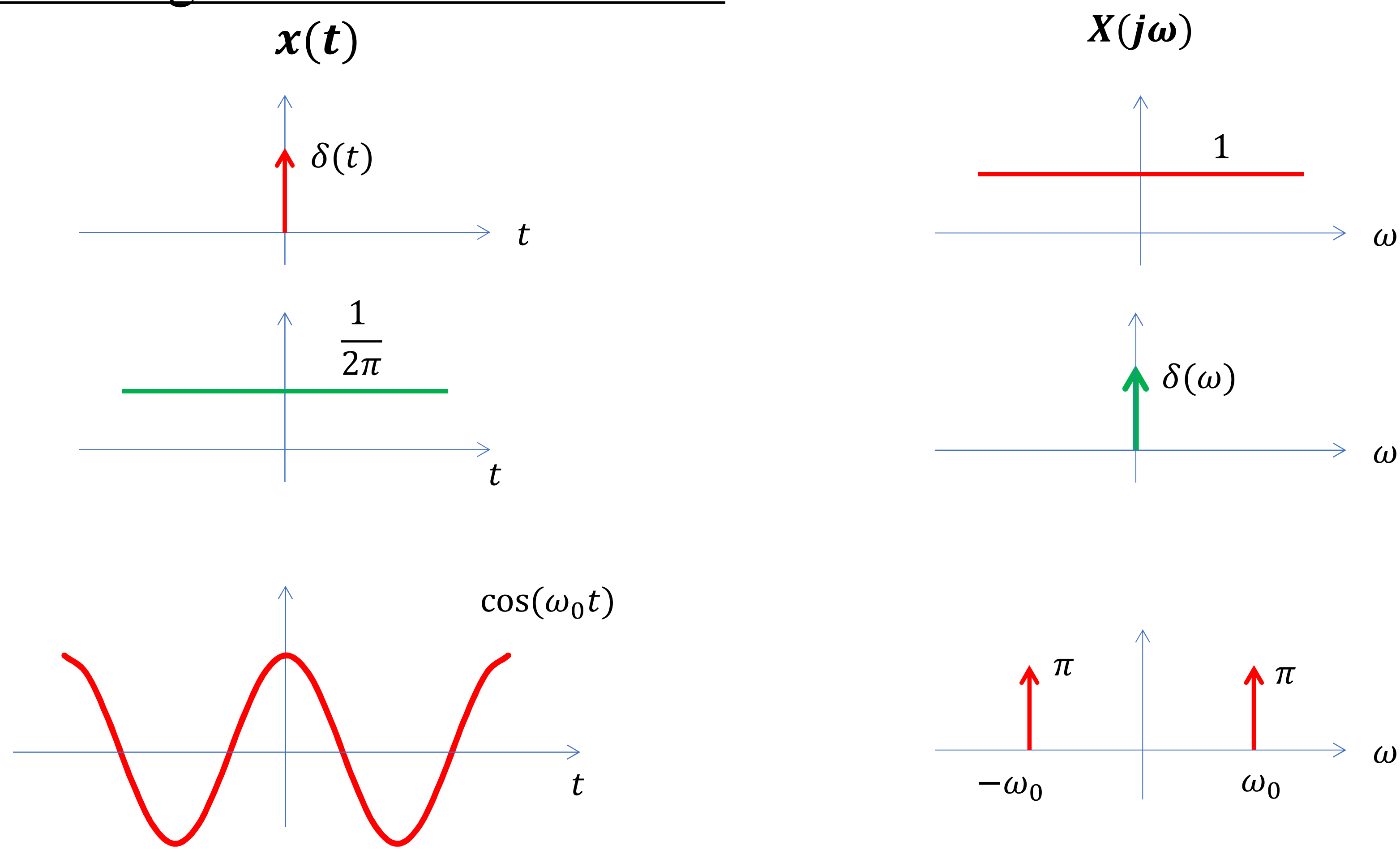
$\delta(t)$ is built by adding periodic spirals of "all" frequencies!

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Delta(j\omega) e^{j\omega t} d\omega$$



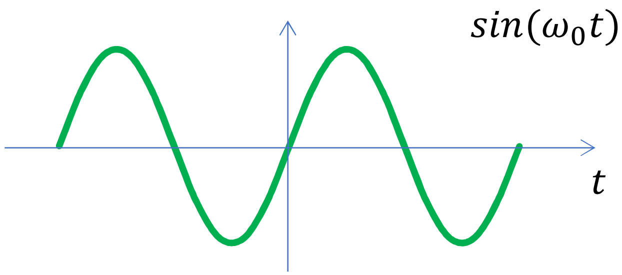


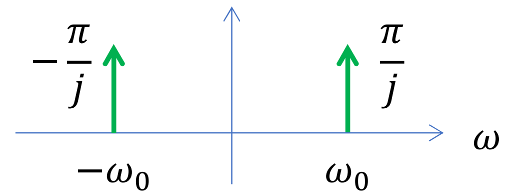
Important Signals Fourier Transforms



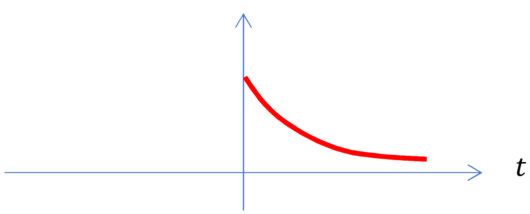
Important Signals Fourier Transforms

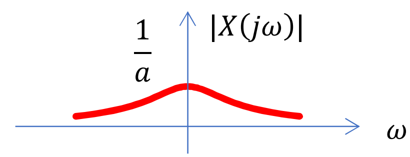
$x(t)$ **$X(j\omega)$**

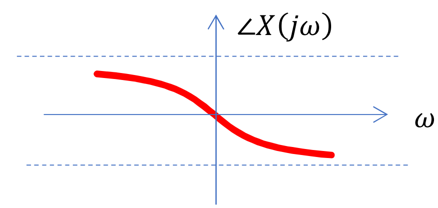




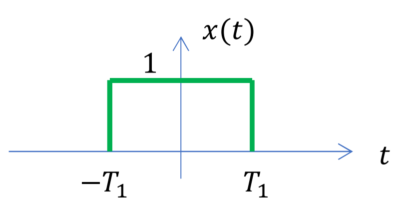
$e^{-a}\ u(t), \qquad a > 0$

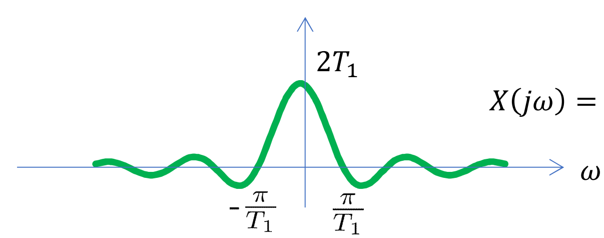






$$X(j\omega) = \frac{1}{j\omega + a}$$





$$X(j\omega) = 2T_1 sinc(\omega T_1)$$

Fourier Transform Properties (Linearity)

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

1. Linearity:

$$x_1(t) \xrightarrow{\text{FT}} X_1(j\omega)$$

$$x_2(t) \xrightarrow{\text{FT}} X_2(j\omega)$$

$$ax_1(t) + bx_2(t) \xrightarrow{\text{FT}} aX_1(j\omega) + bX_2(j\omega)$$

Example:

$$\begin{aligned} x(t) &= \delta(t) + 2e^{-3t}u(t) \\ X(j\omega) &= FT\left(\delta(t)\right) + FT\left(2e^{-3t}u(t)\right) \\ &= 1 + 2FT\left(e^{-3t}u(t)\right) \\ &= 1 + 2\frac{1}{j\omega + 3} \end{aligned}$$

Fourier Transform Properties (Time Shift)

2. Time Shift

$$x_1(t) \xrightarrow{FT} X_1(j\omega)$$

$$x_2(t) = x_1(t - t_0) \xrightarrow{FT} X_2(e^{j\omega} = e^{-j\omega t_0} X_1(j\omega)$$

Delay in time by t_0 is a phase shift by ωt_0 in Fourier transform domain.

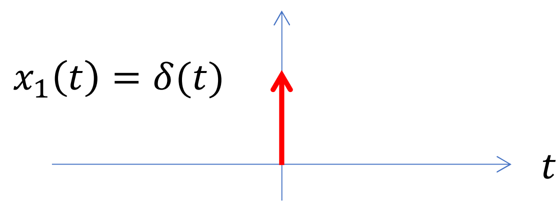
Proof:

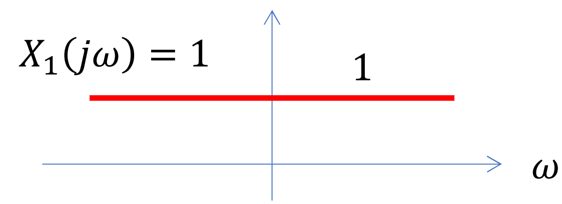
$$\begin{aligned}
X_2(j\omega) &= \underbrace{\int_{-\infty}^{\infty} x_2(t) e^{-j\omega t} dt}_{\text{definition of FT for } x_2(t)} \\
&= \int_{-\infty}^{\infty} x_1(\underbrace{t - t_0}_{\text{change of variable to } u}) e^{-j\omega t} dt \\
&= \int_{-\infty}^{\infty} x_1(u) e^{-j\omega(u + t_0)} du, \quad t - t_0 = u \rightarrow t = u + t_0 \rightarrow du = dt \\
&= \int_{-\infty}^{\infty} x_1(u) e^{-j\omega u} e^{-j\omega t_0} du \\
&= e^{-j\omega t_0} \underbrace{\int_{-\infty}^{\infty} x_1(u) e^{-j\omega u} du}_{\text{FT of } x_1(t)} \\
&= e^{-j\omega t_0} X_1(j\omega)
\end{aligned}$$

Note that boundaries of the integral also have to be adjusted. Here:
$t \rightarrow \infty$ then $u \rightarrow \infty$
similarly
$t \rightarrow -\infty$ then $u \rightarrow -\infty$
so the boundaries stay the same

Fourier Transform Properties (Time Shift)

Example:



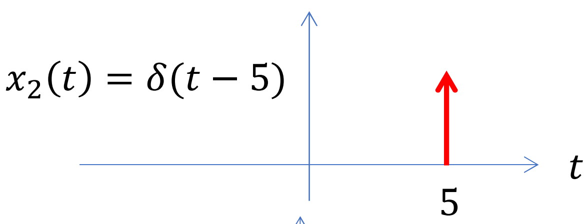


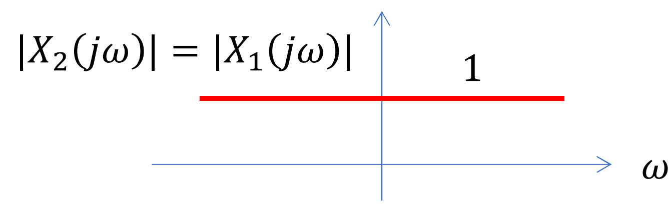
$$x_1(t) = \delta(t) \Rightarrow \quad X_1(j\omega) = 1$$

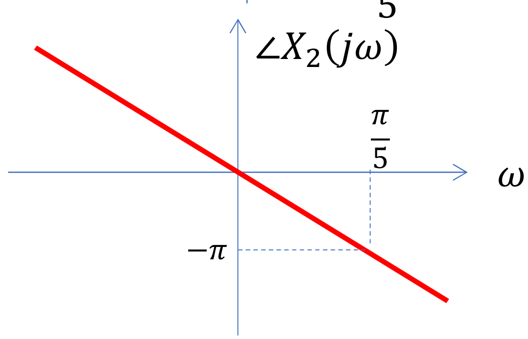
$$x_2(t) = \delta(t-5) = x_1(t-5) \Rightarrow \quad X_2(j\omega) = e^{-j5\omega} X_1(j\omega) = e^{-j5\omega}$$

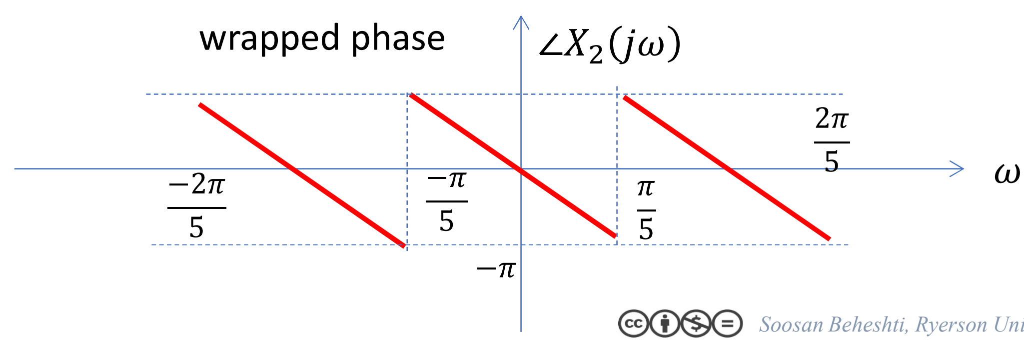
$$|X_2(j\omega)| = |e^{-j5\omega}| = 1$$

$$\angle X_2(j\omega) = \angle X_1(j\omega) - 5\omega = 0 - 5\omega$$



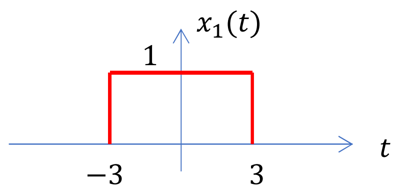


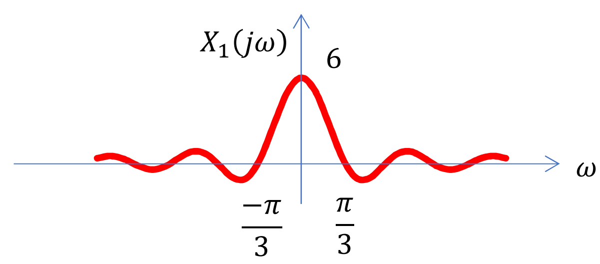


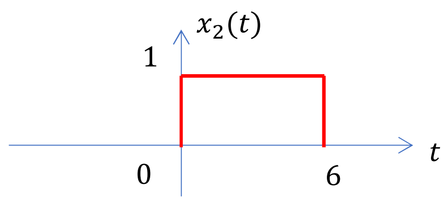


Fourier Transform Properties (Time Shift)

Examples: For the given signal $x(t)$ and its related Fourier transform $X_1(j\omega) = \frac{2sin(3\omega)}{\omega}$, what is FT of $x_2(t)$?







?

$$x_1(t) \xrightarrow{FT} X_1(j\omega)$$

$$x_1(t-t_0) \xrightarrow{FT} e^{-j\omega t_0}X_1(j\omega)$$

Fourier Transform Properties (Time Shift)

Examples: For the given signal $x(t)$ and its related Fourier transform $X_1(j\omega) = \frac{2sin(3\omega)}{\omega}$, what is FT of $x_2(t)$?

$$x_2(t) = x_1(t-3)$$

$$X_2(j\omega) = X_1(j\omega)e^{-j\omega 3} = e^{-j\omega 3}\left(\frac{2sin(3\omega)}{\omega}\right)$$

$$|X_2(j\omega) = |X_1(j\omega)|$$

$$\angle X_2(j\omega) = \angle X_1(j\omega) - 3\omega$$

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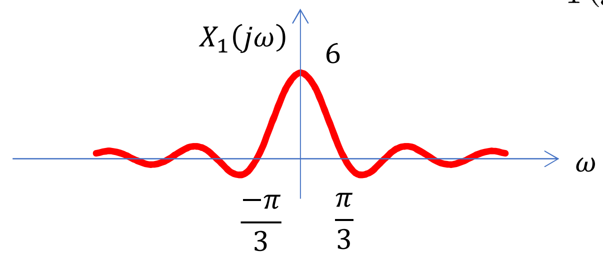

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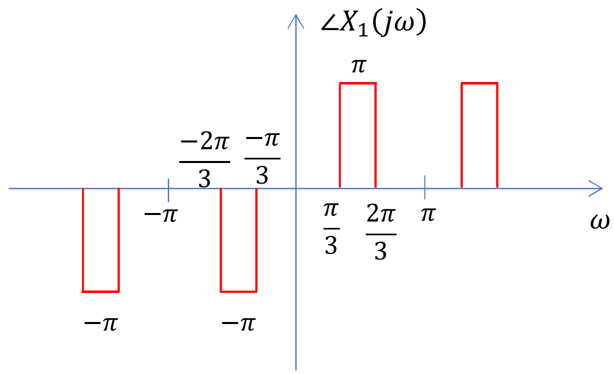

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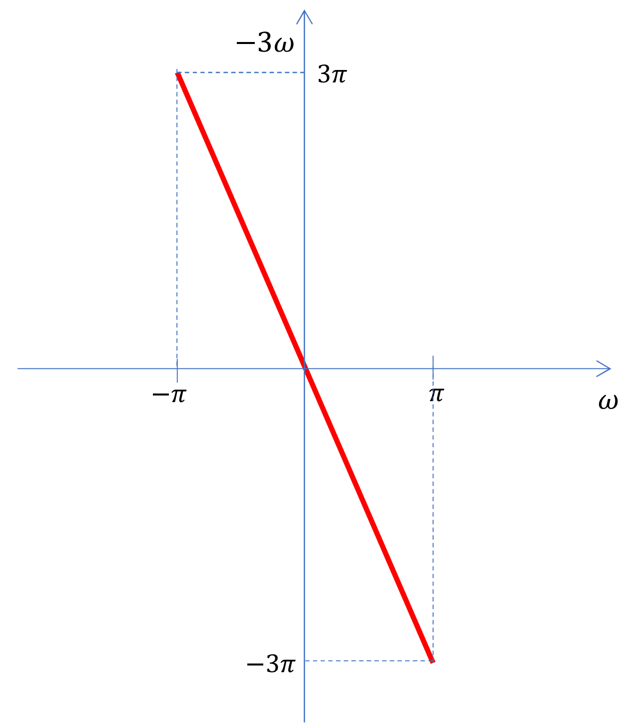
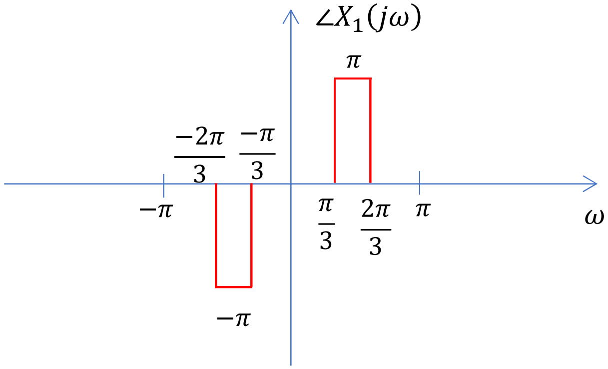


Fourier Transform Properties (Time Shift)



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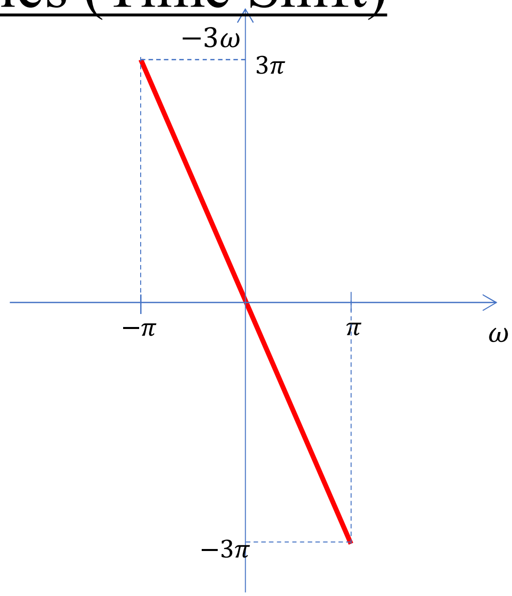

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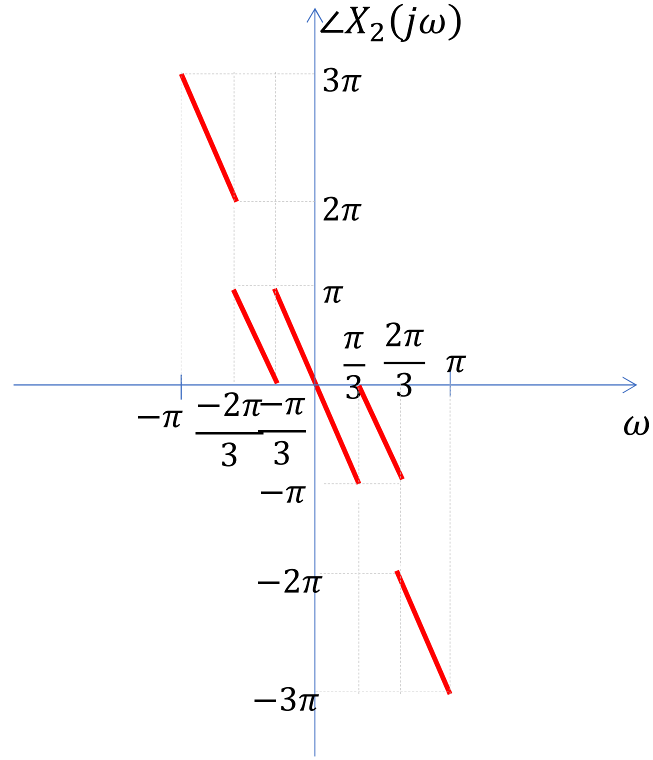

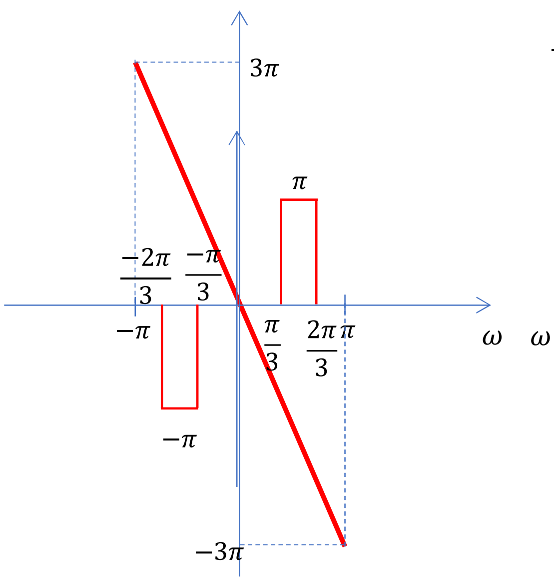

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Fourier Transform Properties (Frequency Shift)

3. Frequency Shift

$$x_1(t) \xrightarrow{FT} X_1(j\omega)$$

$$x_2(t) =? \xrightarrow{FT} X_2(j\omega) = X_1\left(j(\omega - \omega_0)\right)$$

Use change of variable similar to the time shift proof.

$$x_2(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_2(j\omega) e^{j\omega t} d\omega$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Fourier Transform Properties (Frequency Shift)

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$$x_1(t) \xrightarrow{FT} X_1(j\omega)$$

$$x_2(t) =? \xrightarrow{FT} X_2(j\omega) = X_1\left(j(\omega - \omega_0)\right)$$

$$\begin{aligned}
x_2(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_2(j\omega)e^{j\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1\left(j(\omega - \omega_0)\right)e^{j\omega t}d\omega, \quad \omega - \omega_0 = V \rightarrow \omega = V + \omega_0 \rightarrow dV = d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(jV)e^{j(V+\omega_0)t}dV \\
&= e^{j\omega_0 t} \times \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(jV)e^{jVt}dV \\
&= e^{j\omega_0 t}x_1(t)
\end{aligned}$$

Fourier Transform Properties (Frequency Shift)

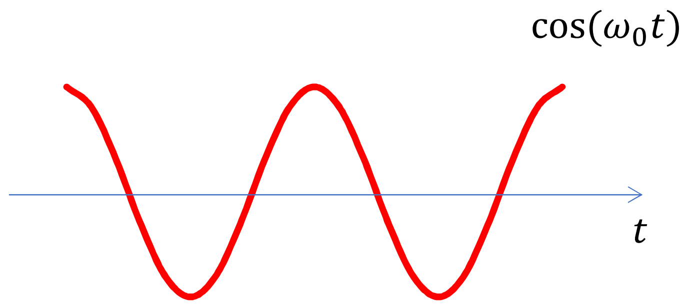
Example:

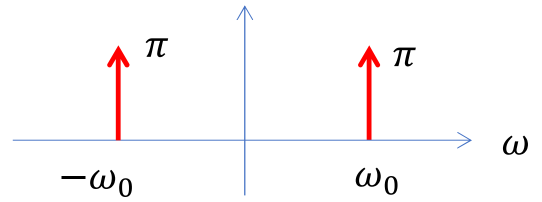
$$x_1(t) = \frac{1}{2\pi} \quad \xrightarrow{IFT} \quad X_1(j\omega) = \delta(\omega)$$

$$X_2(j\omega) = X_1\left(j(\omega - \omega_0)\right) = \delta(\omega - \omega_0) \xrightarrow{IFT} x_2(t) = \frac{1}{2\pi}e^{j\omega_0 t}$$

$$X_3(j\omega) = X_1\left(j(\omega + \omega_0)\right) = \delta(\omega + \omega_0) \xrightarrow{IFT} x_3(t) = \frac{1}{2\pi}e^{-j\omega_0 t}$$

$$X_2(j\omega) + X_3(j\omega) \xrightarrow{IFT} \pi\left(x_2(t) + x_3(t)\right) = \cos(\omega_0 t)$$





Fourier Transform Properties (Frequency Shift)

Example:

$$x_1(t) = e^{-3t}u(t) \longrightarrow \quad X_1(j\omega) = \frac{1}{j\omega + 3}$$

$$x_2(t) = e^{j\frac{\pi}{2}t}e^{-3t}u(t) \longleftarrow \quad X_2(j\omega) = X_1\left(j(\omega - \underbrace{\frac{\pi}{2}}_{\omega_0})\right) = \frac{1}{j(\omega - \frac{\pi}{2}) + 3}$$

$$x_3(t) = e^{-j\frac{\pi}{2}t}e^{-3t}u(t) \longleftarrow \quad X_3(j\omega) = X_1\left(j(\omega + \underbrace{\frac{\pi}{2}}_{\omega_0 = -\frac{\pi}{2}})\right) = \frac{1}{j(\omega + \frac{\pi}{2}) + 3}$$

$$x_4(t) = x_2(t) + x_3(t) = e^{-3t}u(t)\left(e^{j\frac{\pi}{2}t} + e^{-j\frac{\pi}{2}t}\right) = \underbrace{e^{-3t}u(t) \times 2\cos(\frac{\pi}{2}t)}_{\text{Amplitude Modulation (AM) in time}}$$

Fourier Transform Properties (Frequency Shift)

Example:

$$x_1(t) = e^{-3t}u(t) \longrightarrow \quad X_1(j\omega) = \frac{1}{j\omega + 3}$$

$$x_2(t) = e^{j\frac{\pi}{2}t}e^{-3t}u(t) \longleftarrow \quad X_2(j\omega) = X_1\left(j(\omega - \underbrace{\frac{\pi}{2}}_{\omega_0})\right) = \frac{1}{j(\omega - \frac{\pi}{2}) + 3}$$

$$x_3(t) = e^{-j\frac{\pi}{2}t}e^{-3t}u(t) \longleftarrow \quad X_3(j\omega) = X_1\left(j(\omega + \underbrace{\frac{\pi}{2}}_{\omega_0 = -\frac{\pi}{2}})\right) = \frac{1}{j(\omega + \frac{\pi}{2}) + 3}$$

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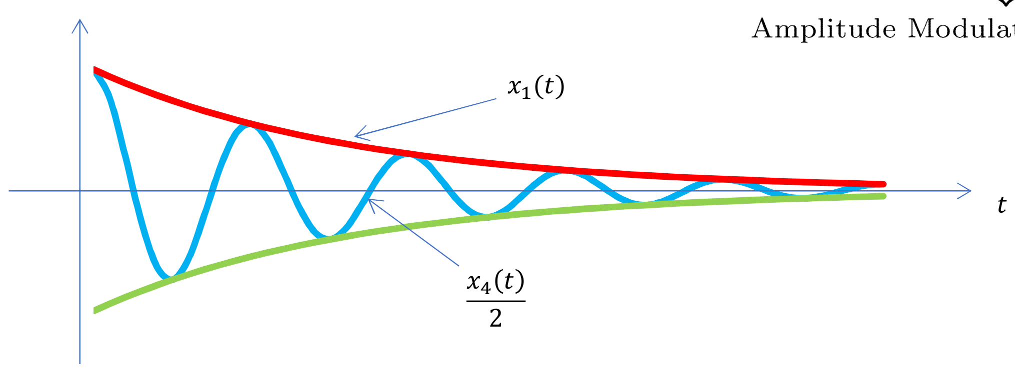
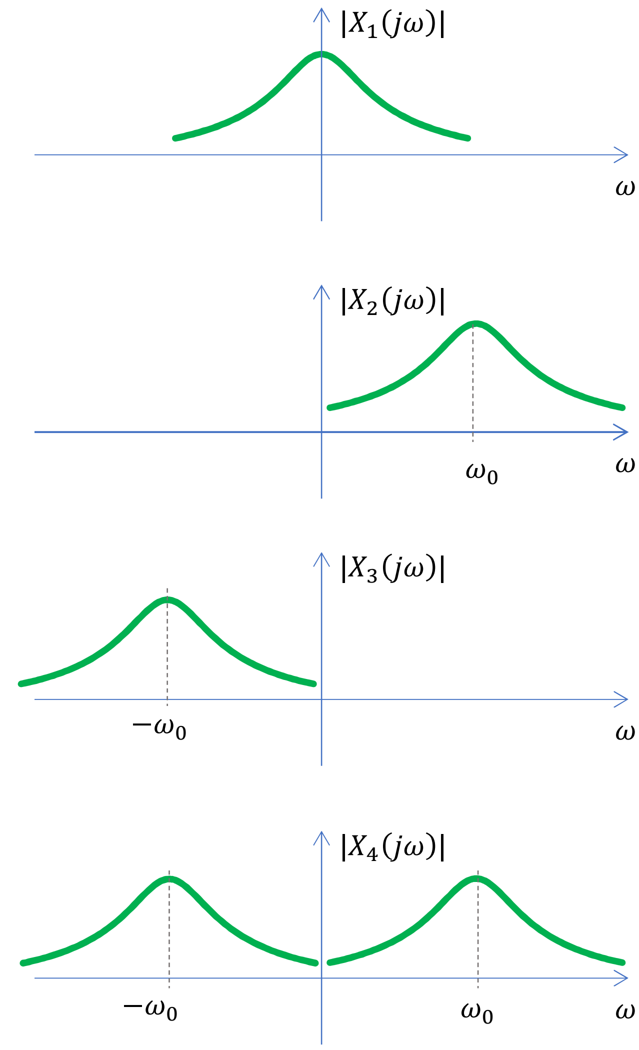




Fourier Transform Properties (Frequency Shift)

Fourier Transform of causal part of real part of an exponential signal ($\sigma > 0$):

$$x(t) = e^{-\sigma t}\cos(\omega_0 t)u(t)$$

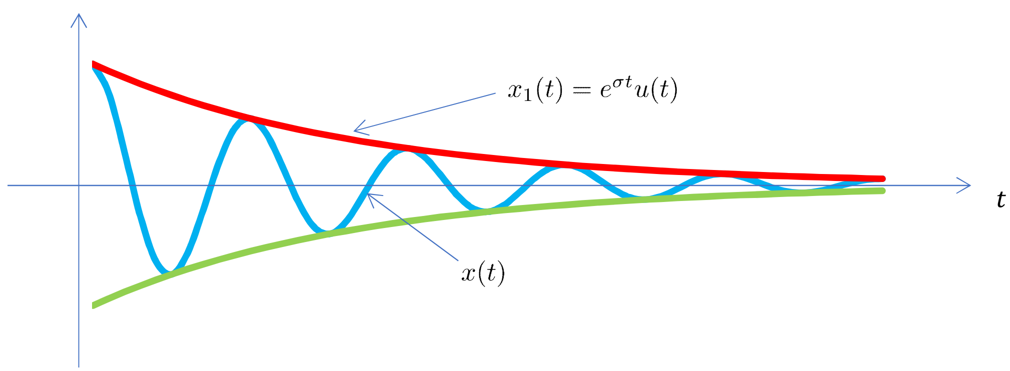
$$x_1(t) = e^{-\sigma t}u(t) \qquad FT(x_1(t)) = X_1(j\omega) = \frac{1}{j\omega + \sigma}$$

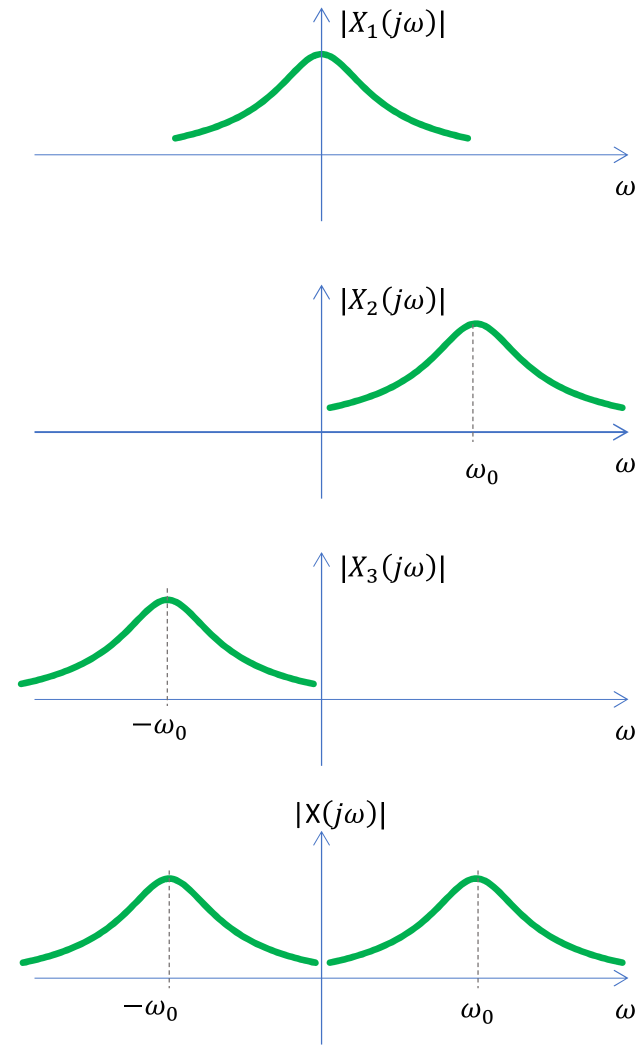
$$x(t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}x_1(t)$$

$$FT(x(t)) = \frac{1}{2}(X_1(j\omega - \omega_0) + X_1(j\omega + \omega_0))$$

$$= \frac{1}{2}\left(\frac{1}{j(\omega - \omega_0) + \sigma} + \frac{1}{j(\omega + \omega_0) + \sigma}\right) = \frac{1}{2}\left(\frac{1}{j\omega + \sigma - j\omega_0} + \frac{1}{j\omega + \sigma + j\omega_0}\right)$$

$$= \frac{1}{2}\frac{j\omega + \sigma - j\omega_0 + j\omega + \sigma + j\omega_0}{(j\omega + \sigma)^2 - (j\omega_0)^2} = \frac{j\omega + \sigma}{(j\omega + \sigma)^2 + \omega_0^2}$$





Fourier Transform Properties (Time Scaling)

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

4. Scaling

$$x_1(t) \xrightarrow{FT} X_1(j\omega)$$

$$x_2(t) = x_1(at) \xrightarrow{FT} X_2(j\omega) =? \quad \text{as a function of} \quad X_1(j\omega)$$

Use change of variable similar to the previous properties

$$X_2(j\omega) = \int_{-\infty}^{\infty} x_2(t)e^{-j\omega t}dt$$

Fourier Transform Properties (Time Scaling)

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

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$$x_1(t) \xrightarrow{FT} X_1(j\omega)$$

$$x_2(t) = x_1(at) \xrightarrow{FT} X_2(j\omega) = ? \quad \text{as a function of } \; X_1(j\omega)$$

$$\begin{aligned}
X_2(j\omega) &= \int_{-\infty}^{\infty} x_2(t)e^{-j\omega t}dt \\
&= \int_{-\infty}^{\infty} x_1(at)e^{-j\omega t}dt, \quad v = at \rightarrow t = \frac{v}{a} \rightarrow dv = adt \\
\text{If } a > 0 \quad &= \int_{-\infty}^{\infty} x_1(v)e^{-j\omega\frac{v}{a}}\frac{dv}{a} = \frac{1}{a}\int_{-\infty}^{\infty} x_1(v)e^{-j\frac{\omega}{a}v}dv = \frac{1}{a}X_1(j\frac{\omega}{a}) \\
\text{If } a < 0 \quad &= \frac{1}{|a|}\int_{-\infty}^{\infty} x_1(v)e^{-j\frac{\omega}{a}v}dv = \frac{1}{|a|}X_1(j\frac{\omega}{a})
\end{aligned}$$

For $a < 0$ we have $dt = -\frac{dv}{|a|}$,
therefore
$t \rightarrow \infty$ then $u \rightarrow -\infty$
similarly
$t \rightarrow -\infty$ then $u \rightarrow \infty$

So we have $\int_{\infty}^{-\infty} \cdots dv = -\int_{-\infty}^{\infty} \cdots dv$

In general for all a

$$X_2(j\omega) = \frac{1}{|a|}X_1(j\frac{\omega}{a})$$

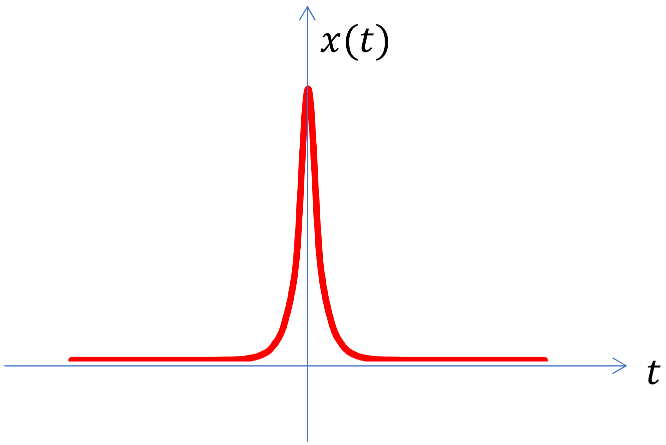
If $a = -1$

$$x(-t) \longrightarrow X(-j\omega)$$

Fourier Transform Properties (Scaling)

Example:

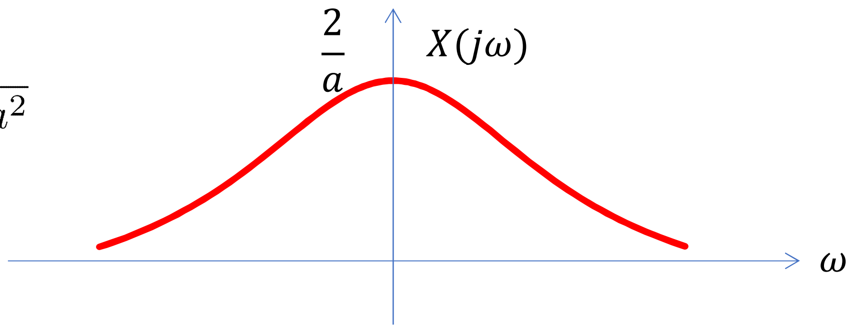
$$x(t) = e^{-2|t|}$$
$$= \underbrace{e^{-2t}u(t)}_{x_1(t)} + \underbrace{e^{2t}u(-t)}_{x_1(-t)}$$



$$X(j\omega) = X_1(j\omega) + X_1(-j\omega)$$
$$= \frac{1}{j\omega + 2} + \frac{1}{-j\omega + 2}$$
$$= \frac{2 \times 2}{\omega^2 + 4} = \frac{4}{\omega^2 + 4}$$

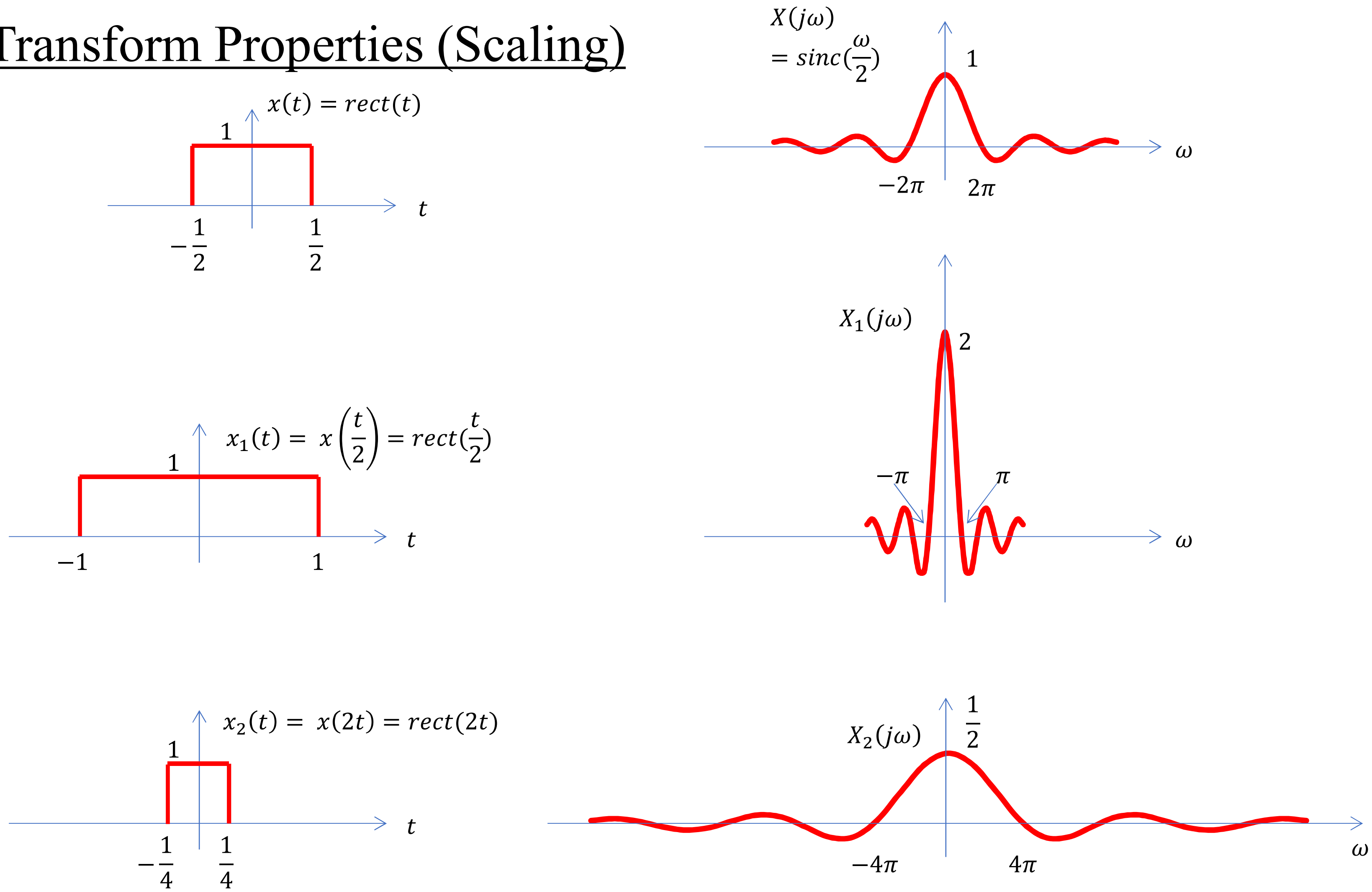
In general:

$$x(t) = e^{-|a|t} \longrightarrow X(j\omega) = \frac{2a}{\omega^2 + a^2}$$



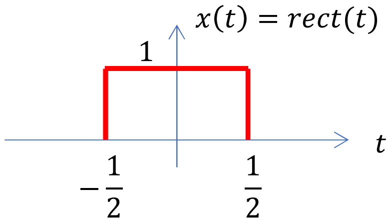
Fourier Transform Properties (Scaling)

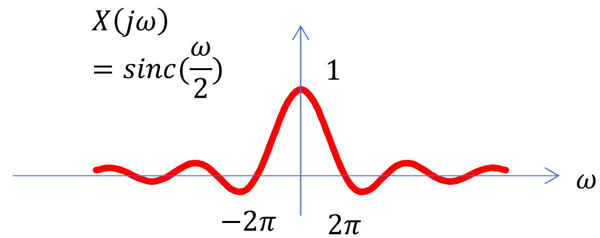
Example:

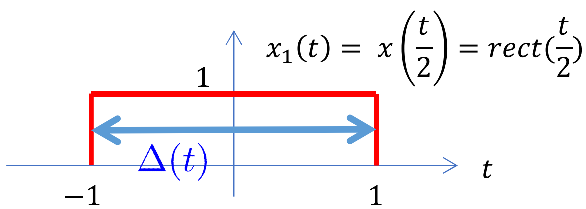


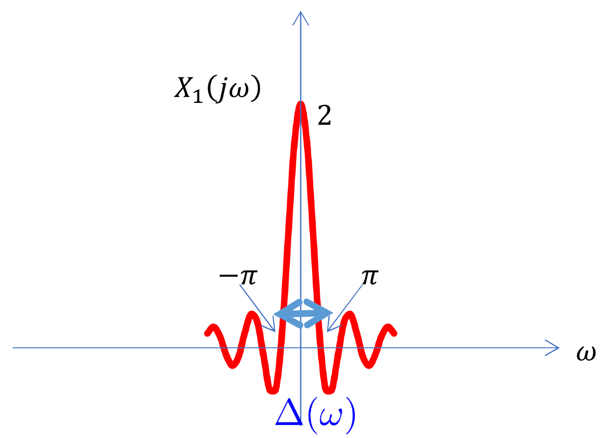
Fourier Transform Properties (Scaling)

Example:

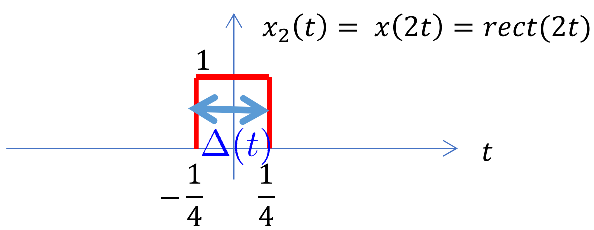


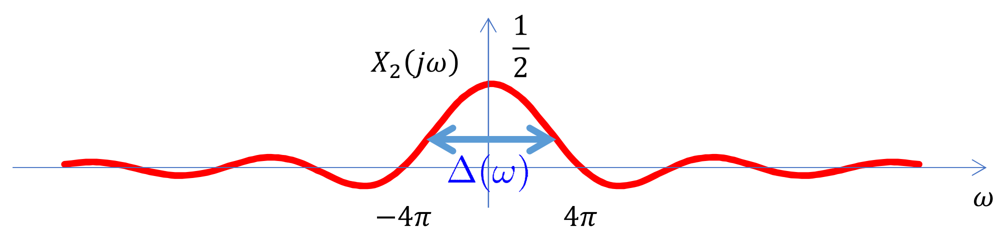






This is consistent with **Heisenberg's uncertainty principle:**
$\Delta t \times \Delta\omega > constant\ value$





Fourier Transform Properties (Duality)

5. Duality

$$x_1(t) \longrightarrow X_1(j\omega)$$

$$x_2(t) = X_1(jt) \longrightarrow X_2(j\omega) =?$$

$$X_2(j\omega) = \int_{-\infty}^{\infty} x_2(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} X_1(jt)e^{-j\omega t}dt$$

Replace ω with V

$$x_1(t) = \frac{1}{2\pi}\int X_1(j\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int X_1(jV)e^{jVt}dV$$

Replace t with ω

$$x_1(\omega) = \frac{1}{2\pi}\int X_1(jV)e^{jV\omega}dV$$

Replace V with t

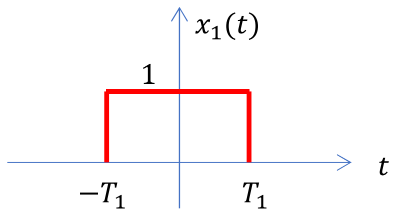
$$2\pi x_1(-\omega) = \int X_1(jt)e^{-jt\omega}dt$$

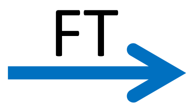
$$2\pi x_1(-\omega) = X_2(j\omega)$$

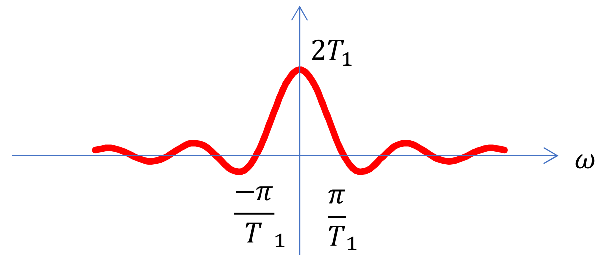
Fourier Transform Properties (Duality)

$$X_1(j\omega) = 2T_1 \frac{\sin(\omega T_1)}{\omega T_1}$$

Example:

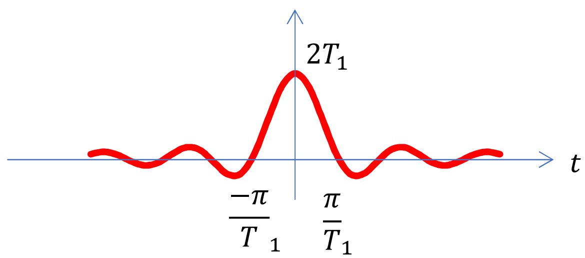


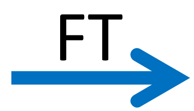


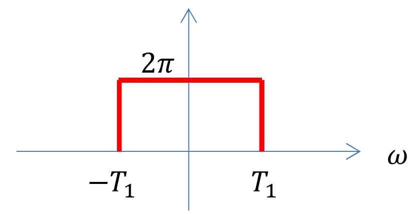


$$x_2(t) = X_1(jt) = 2T_1 \frac{\sin(tT_1)}{tT_1}$$

$$X_2(j\omega) = 2\pi x_1(-\omega)$$







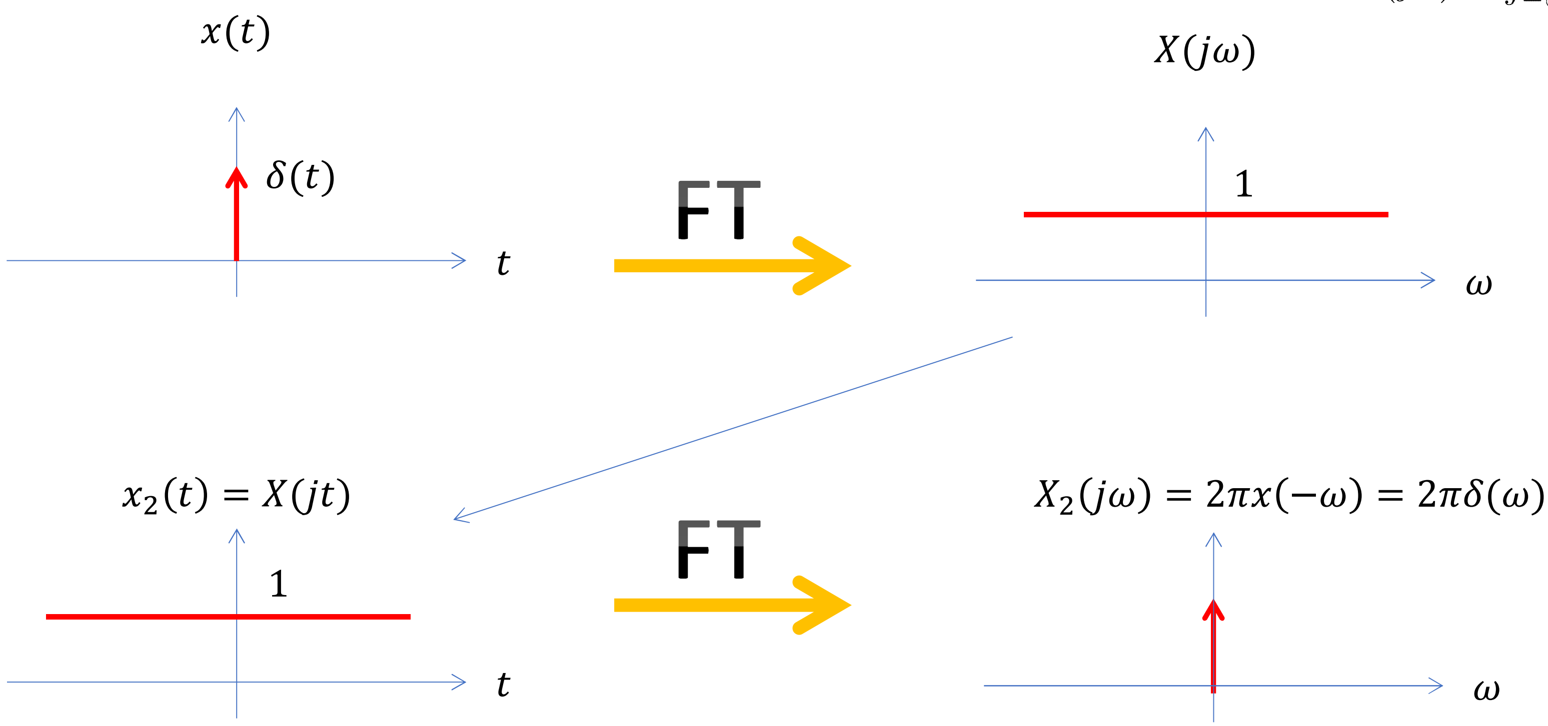
Check this answer:

$$x_2(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_2(j\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-T_1}^{T_1} 2\pi e^{j\omega t}d\omega$$
$$= \frac{e^{jtT_1} - e^{-jtT_1}}{jt} = \frac{2sin(tT_1)}{t}$$

Fourier Transform Properties (Duality)

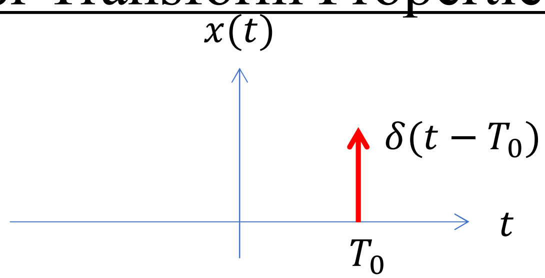
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$



Fourier Transform Properties (Duality)

Example:



FT →

Periodic Spiral in Frequency domain with freq. T_0

$$X(j\omega) = e^{-j\omega T_0}$$

Periodic Spiral in Time with freq. T_0

$$x_2(t) = e^{-jT_0 t}$$

FT →

$$X_2(j\omega) = 2\pi \overbrace{\delta(-\omega - T_0)}^{x(-\omega)}$$

$$= 2\pi\delta(\omega + T_0)$$

Same for shift to other direction:

$$e^{jT_0 t} \xrightarrow{FT} 2\pi\delta(\omega - T_0)$$

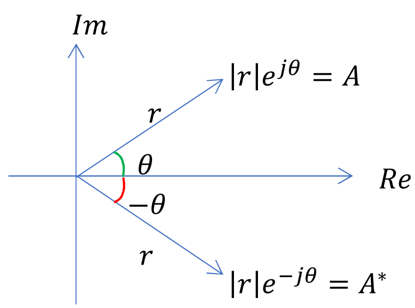
Fourier Transform Properties (Conjugate property)

6. Conjugate Property:

$$x(t) \xrightarrow{FT} X(j\omega)$$

$$x^*(t) \xrightarrow{FT} X^*(-j\omega)$$

Reminder:



$A = a + jb$
$A^* = a - jb$

complex conjugate of a real number is itself!

Example:

$$x(t) = e^{j\omega_0 t} \longrightarrow \quad 2\pi\delta(\omega - \omega_0)$$

$$x^*(t) = (e^{j\omega_0 t})^* = e^{-j\omega_0 t} \longrightarrow X^*(-j\omega) = (2\pi\delta(-\omega - \omega_0))^* = 2\pi\delta(-\omega - \omega_0) = 2\pi\delta(\omega + \omega_0)$$

Conjugate symmetry property for real signals

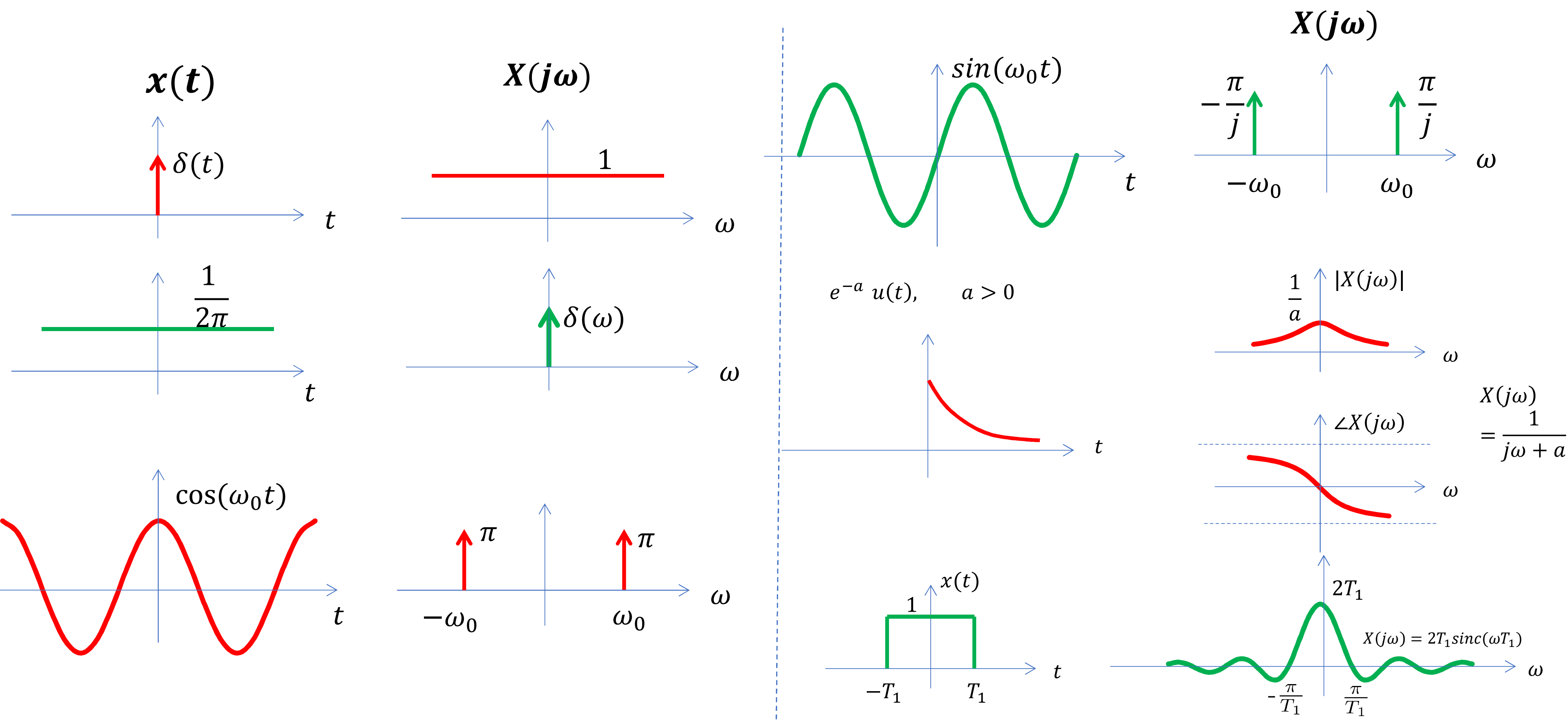
If $x(t)$ is real: $x^*(t) = x(t)$ therefore:

$$FT(x^*(t)) = FT(x(t))$$
$$X^*(-j\omega) = X(j\omega)$$
$$|X(-j\omega)|e^{-j\angle X(-j\omega)} = |X(j\omega)|e^{j\angle X(j\omega)}$$

For real signals:

1- $|X(-j\omega)| = |X(j\omega)|$ Absolute value is an even function
2- $-\angle X(-j\omega) = \angle X(j\omega)$ Phase is an odd function

Complex Conjugate property for real signals



Fourier Transform & Convolution

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

7. FT & Convolution

$$x_1(t) \xrightarrow{FT} X_1(j\omega)$$

$$x_2(t) \xrightarrow{FT} X_2(j\omega)$$

$$x_3(t) = x_1(t) * x_2(t) \xrightarrow{FT} X_3(j\omega) = X_1(j\omega) \times X_2(j\omega)$$

Convolution in time $\equiv$ Product in Frequency

Example:

$$x_1(t) = e^{-2t}u(t) \xrightarrow{FT} \frac{1}{j\omega + 2}$$

$$x_2(t) = \delta(t-3) \xrightarrow{FT} e^{-j\omega 3}$$

$$x_1(t) * x_2(t) = e^{-2t}u(t) * \delta(t-3) = e^{-2(t-3)}u(t-3) = e^{6}e^{-2t}u(t-3)$$

FT transform using the product

$$X_1(j\omega) \times X_2(j\omega) = \frac{e^{-j\omega 3}}{j\omega + 2} = X_3(j\omega)$$

Fourier Transform & Convolution

Example:

FT transform using the product

$$X_1(j\omega) \times X_2(j\omega) = \frac{e^{-j\omega 3}}{j\omega + 2} = X_3(j\omega)$$

FT transform using the definition

$$\begin{aligned}
X_3(j\omega) &= \int_{-\infty}^{\infty} x_3(t) e^{-j\omega t} dt \\
&= \int_{-\infty}^{\infty} e^6 e^{-2t} u(t-3) e^{-j\omega t} dt \\
&= e^6 \int_{3}^{\infty} e^{-2t} e^{-j\omega t} dt \\
&= e^6 \frac{e^{-t(2+j\omega)}}{-(2+j\omega)} \bigg|_3^{\infty} \\
&= -e^6 \frac{e^{-3(2+j\omega)}}{-(2+j\omega)} \\
&= \frac{e^{-3j\omega}}{2+j\omega}
\end{aligned}$$

Fourier Transform & Periodic Signals

8. FT & periodic signals

$$x_p(t) = \sum_n D_n e^{j\omega_0 nt} \xrightarrow{FT} X_p(j\omega) = FT\left(\sum D_n e^{j\omega_0 nt}\right)$$

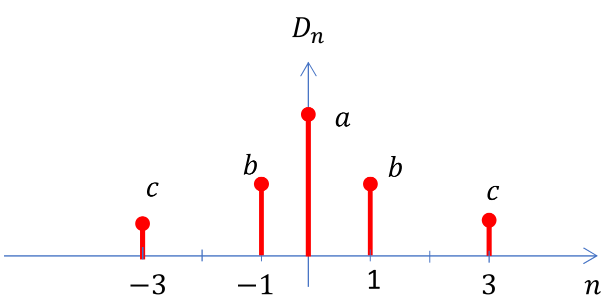
$$= \sum FT\left(D_n e^{j\omega_0 nt}\right)$$

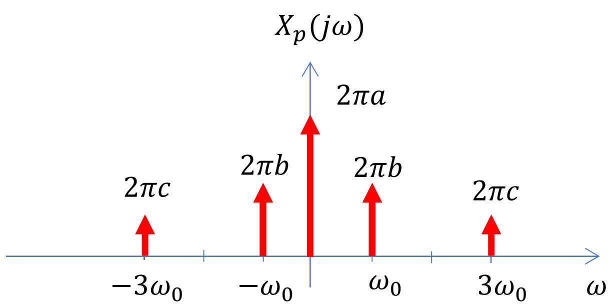
$$= \sum D_n FT\left(e^{j\omega_0 nt}\right)$$

$$= \sum D_n 2\pi\delta\left(\omega - \omega_0 n\right)$$

To calculate FT of a periodic signal
first find its FS and then use this property.

$$X_p(j\omega) = \sum_n D_n 2\pi\delta(\omega - \omega_0 n)$$





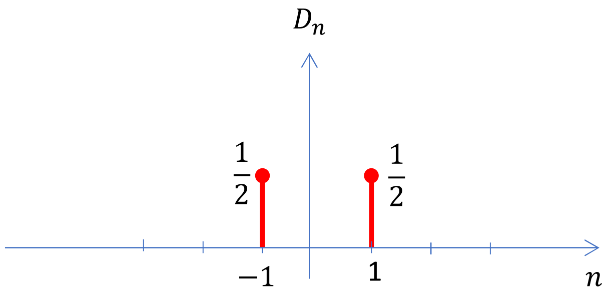
Fourier Transform & Periodic Signals

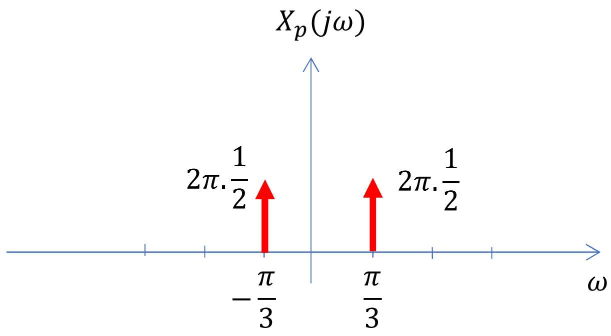
Example:

$$x_p(t) = cos(\underbrace{\frac{\pi}{3}}_{\omega_0} t) = \underbrace{\frac{1}{2}}_{D_1} e^{j\frac{\pi}{3}t} + \underbrace{\frac{1}{2}}_{D_{-1}} e^{-j\frac{\pi}{3}t}$$

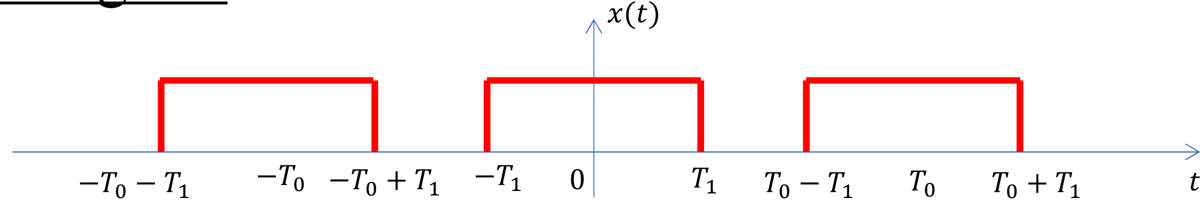
$$X_p(j\omega) = \pi\delta(\omega - \underbrace{\omega_0}_{\omega_0=\frac{\pi}{3}}) + \pi\delta(\omega + \omega_0)$$

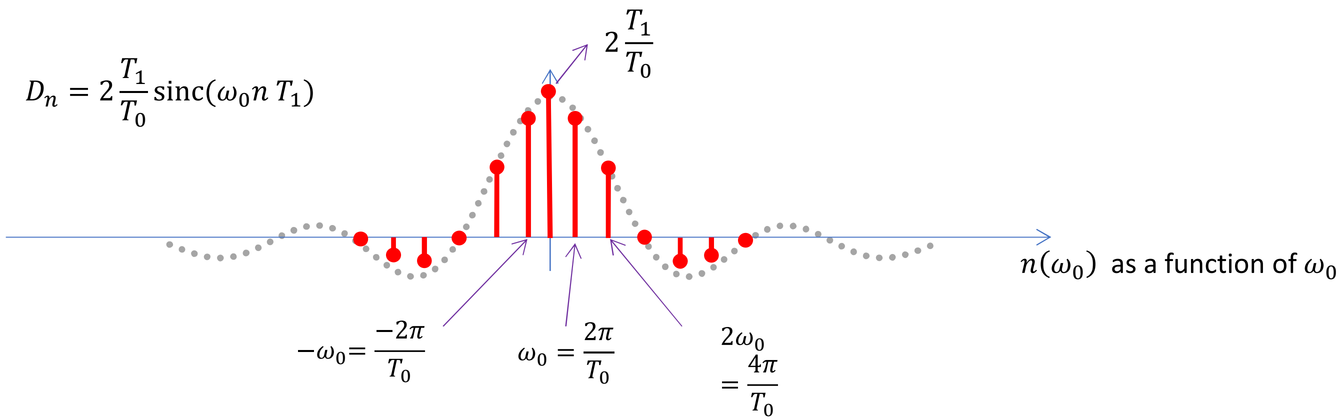
There are two methods to find ft of sin and cos through shift and connection of periodic signals.

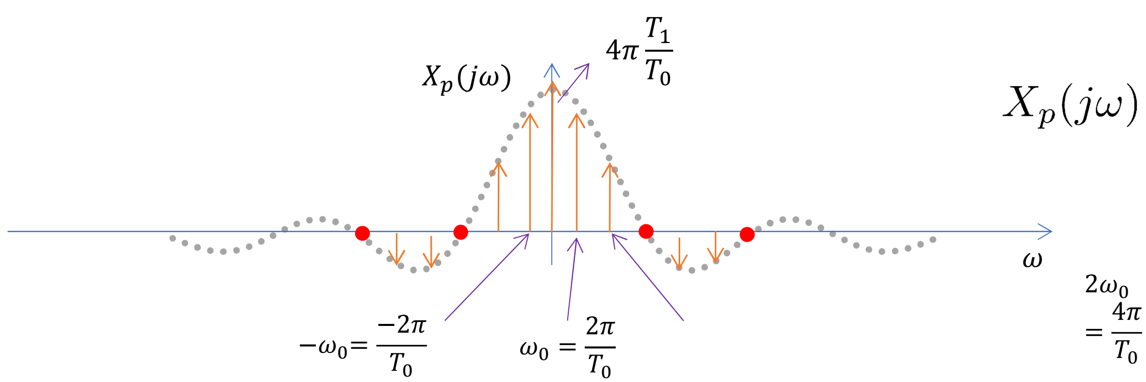




Fourier Transform & Periodic Signals







$$X_p(j\omega) = \sum 2\pi \left(\frac{2T_1}{T_0} sinc(\omega_0 n T_1) \right) \delta(\omega - n\omega_0)$$

Fourier Transform & Periodic Signals

FT properties	Signal	FT
	$x(t)$	$X(j\omega)$
	$z(t)$	$Z(j\omega)$
Linearity	$ax(t) + bz(t)$	$aX(j\omega) + bZ(j\omega)$
Time shift	$x(t - T_0)$	$e^{-j\omega T_0} X(j\omega)$
Freq. shift	$e^{j\omega_0 t} x(t)$	$X(j(\omega - \omega_0))$
Scaling	$x(at)$	$\frac{1}{\vert a\vert} X(j\frac{\omega}{a})$
Duality	$X(jt)$	$2\pi x(-\omega)$
Complex Conj.	$x^*(t)$	$X^*(-j\omega)$ (so for real signals $\vert X(j\omega\vert$ is even and $\angle(X(j\omega)$ is odd
Convolution	$x(t) * z(t)$	$X(j\omega) \times Z(j\omega)$
Periodic signals	$x_p(t)$ with D_n coeffs	$X_p(j\omega) = \sum_n D_n 2\pi\delta(\omega - \omega_0 n)$