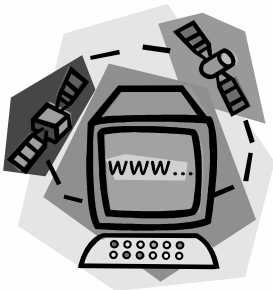
EE 8107 Digital Communications

Course Notes



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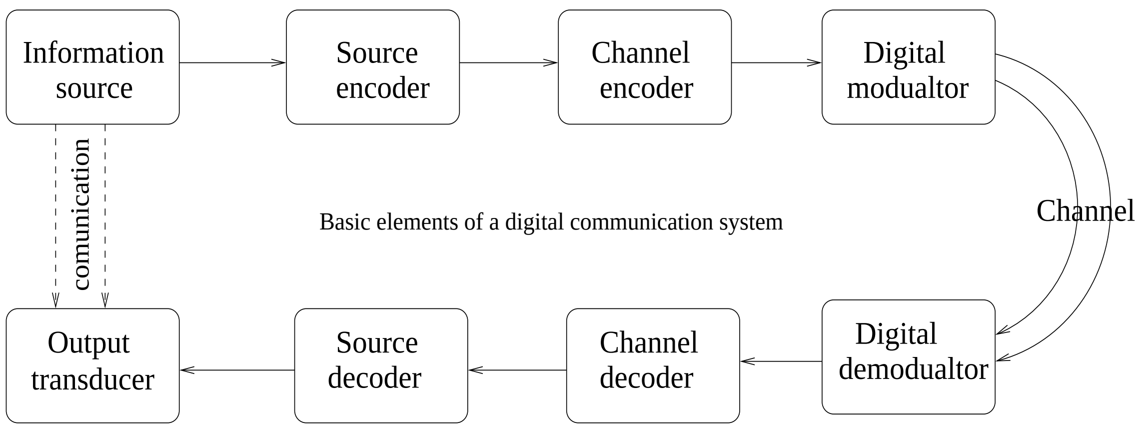
1 Overview of a Digital Communication System

- Any communication systems involve, source ⇒ sink via a channel.
- Source (or sink) can be analog (e.g., audio) or digital (e.g., fax).
- In digital communication, a sequence of binary digits (discrete in time and finite number of them) is transmitted.
- Goal: **efficient** and **reliable** communication over **noisy channel**.



Signal bandwidth: The (base-band) signal is said to have bandwidth of B Hz if most (e.g., 95%) of the energy of the signal is contained in frequency components smaller than B Hz. And, pass-band DSB signals have twice the bandwidth of the base-band signals.

Coders: Source coder does compression for efficient transmission (i.e. removes redundancy) and channel coder prepares for reliable transmission over the noisy channel (i.e. may increase data rates for error detection/correction.)

Digital modulator: It is an interface to the channel, i.e., (digital) 0101011 to (analog) waveforms, e.g., modem. *M-ary modulation:* Take a b-bit sequence and transmit one of $M = 2^b$ possible waveforms (a *symbol* is transmitted).

♠ Example with b = 2.

Rate is b-fold decreased. How about average power and bandwidth ?.

Communication channel: Examples are microwave radio, fiber or cable. Thermal noise in the channel corrupts the waveforms and symbol by symbol detection is done to declare the transmitted symbol in digital communication. Multiple access interference (MAI) is also a cause for impairment, for example, wireless cellular systems and Ethernet.

System performance: A digital communication system can be characterized by performance measure such as

1. bit rate (copper wire (kbps), optical fiber (Gbps), cellular radio (kbps))
2. bit error rate (e.g., 10^{-3} in telephony), 10^{-8} in mission-critical apps.
3. propagation and processing delay (important in real-time applications such as audio and video)

   Bit error rate (BER) is dependent on:

   (i) coding techniques (ii) transmitted waveforms (or signals) (iii) transmit power (iv) channel characteristics (v) modulation/demodulation and (vi) receiver design.

Advantages of digital communication: higher capacity, store/forward capability, switching/networking facility, high fidelity and enhanced services using advanced DSP techniques.

2 Communication Channel Models

- Problem arises when the noise and the signal of interest occupy the same frequency band. Unfortunately, noise occupy the entire band!
- A major source of noise is (additive) thermal noise.
- Proper system design (i.e., waveform, modulation, receiver etc.) can minimize the adverse effect of the noise.
- Mathematical models and tools help us understand channel characteristics and design the signal/system accordingly.

Additive Noise Channel: Signal is corrupted by additive noise process $n(t)$. Generally, characterized statistically as additive Gaussian noise process (AWGN) with flat power spectrum density of $\frac{N_o}{2}$.

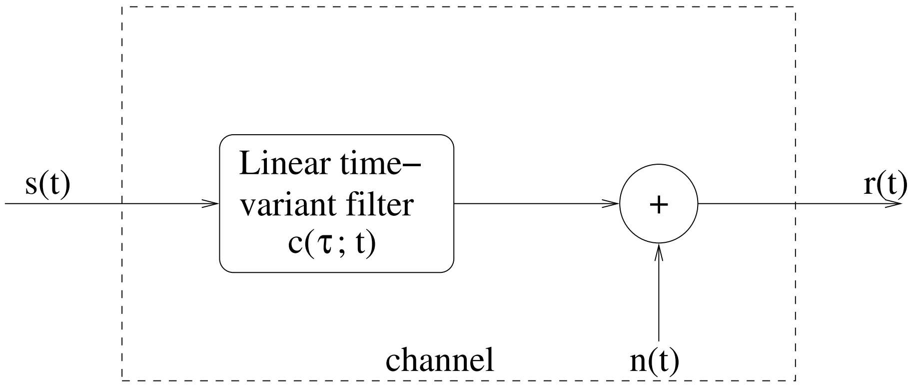
$$r(t) = \alpha s(t) + n(t)$$

Linear Time-invariant Channel: Front-end filters are often used in communication systems to eliminate the out-of-band noise or signal. They are generally characterized as LTI systems.

$$r(t) = s(t) \star c(t) + n(t) = \int_{-\infty}^{+\infty} c(\tau)s(t-\tau)d\tau + n(t)$$

Linear Time-variant Channel: Example: multipath signals that arrive with different delay and attenuation in different environment.

$$r(t) = s(t) \star c(\tau; t) + n(t)$$



where $c(\tau;t)$ is the impulse response of the channel (filter) at time t for an impulse applied at time $t-\tau$.

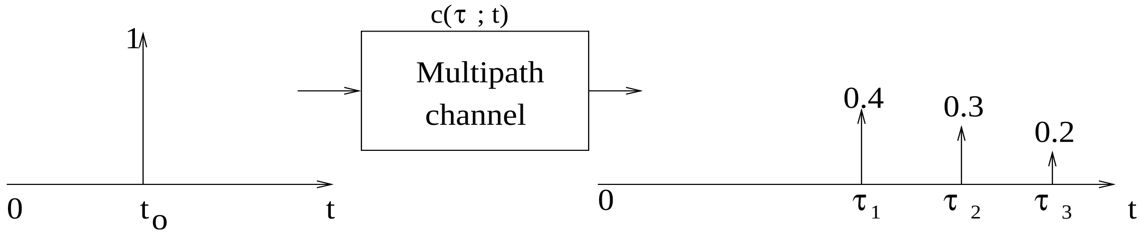
A radio multipath channel is characterized with its impulse response as,

$$c(\tau;t) = \sum_{k=1}^{L} a_k(t)\delta(\tau-\tau_k)$$

where $a_k(t)$ represents time-varying attenuation factor for the L multipaths and τ_k are the corresponding delays. Hence,

$$r(t) = \sum_{k=1}^{L} a_k(t)s(t-\tau_k) + n(t)$$

Received signal consists of L multipath components each characterized by an attenuation and a delay component.



3 Probability Theory

An important mathematical tool in the design and evaluation of digital communication systems. Examples:

1. statistical modelling of information sources (e.g., apriori probability of symbol transmission). The occurrence of A's in a fax message?

2. characterization of communication channels (e.g., corruption of bits in binary symmetric channel)

3. evaluation of different modulation schemes (e.g., probability of bit error)

4. design of optimum receivers (e.g., matched filter in AWGN channel)

Probability as a long term average. E.g., the probability of a person talking more than 3 mins in a phone conversation? or the probability of a bit being corrupted?

3.1 Event Space and Probability

Notion of sample space (S) and event space (A, B).

A. Mutual Exclusion: Two events, A and B are mutually exclusive if $A \cap B = \emptyset$. If $A_i \cap A_j = \emptyset, i \neq j = 1, 2, ...$ The total probability,

$$P(S) = P\left(\bigcup_i A_i\right) = \sum_i P(A_i) = 1$$

It is easy to visualize with Venn diagrams.

B. Joint Events and Probability: Let $A_i, i = 1, 2, ...n$, and $B_j, j = 1, 2, ...m$, then the joint event (A_i, B_j) has, $0 \leq P(A_i, B_j) \leq 1$. If events $A_i, i =$

$1, 2, ...n$ are mutually exclusive, then

$$\sum_{i=1}^{n} P(A_i, B_j) = P(B_j)$$

♠ Example: $A_i \in \{1, 2, ..., 6\}$ and $B_j \in \{H, T\}$.

C. Conditional Probability: Knowledge of the outcome of one (related) event affects the probability of another event.

$$P(A|B) = \frac{P(A, B)}{P(B)}, \qquad P(B) > 0$$

Also important to note: $P(A, B) = P(A|B)P(B) = P(B|A)P(A)$

- If A and B are mutually exclusive, $P(A|B) = 0$
- If A and B are statistically independent, $P(A|B) = P(A)$

D. Bayes' Theorem: If $A_i, i = 1, 2, ..., n$, are mutually exclusive events, and B is any event with $P(B) > 0$, then

$$\boxed{P(A_i|B) = \frac{P(A_i, B)}{P(B)} = \frac{P(B|A_i)P(A_i)}{\sum_{j=1}^{n} P(B|A_j)P(A_j)}}$$

Since $B = B \cap S, P(B) = P(B \cap A_1) + ... + P(B \cap A_n)$ and $P(B \cap A_i) = P(B|A_i)P(A_i)$. Note that $(A \cap B) = (A, B)$

Bayes' theorem is extremely useful. Examples:

- in computing the bit error rate in binary symmetric channel (BSC)
- in deriving the optimal receiver structure with apriori probabilities.

♠ Example - BSC:

Assuming equal probability for input bits,

$$P_e = P(X = 0|Y = 1)$$

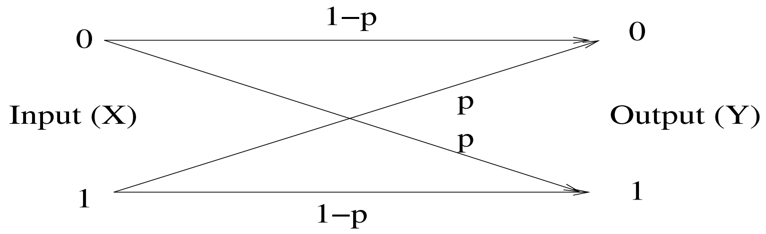


Figure 1: Binary Symmetry Channel

$$P_e = \frac{P(Y=1|X=0)P(X=0)}{P(Y=1|X=0)P(X=0)+P(Y=1|X=1)P(X=1)} = p$$

E. Statistical Independence: Occurrence of A does not depend on the occurrence of B. Hence, $P(A|B) = P(A) \Rightarrow P(A,B) = P(A)P(B)$

In general, $A_j, j = 1, 2, ...n$ are statistically independent, if the *joint probability is the product of the marginal probabilities.*

3.2 Random Variables

A. Definition: Sample space S. Elements, $s \in S$. Function $X(s)$ maps S into real line ($\Re$). $X(s)$ (or conveniently X) is called random variable (RV). Randomness is in s not in X.

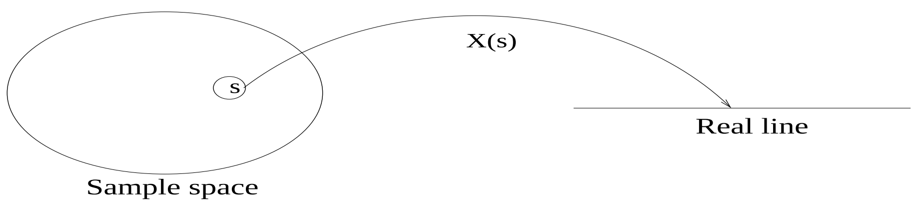


Figure 2: Definition of Random Variable

B. Distributions:

- Cumulative Density Function (CDF): $F(x) = P(X \leq x), x \in \Re$
- Probability Density Function (PDF): $p(x) = \frac{dF(x)}{dx}, x \in \Re$

C. Multiple RVs:

$$F(x_1, x_2) = P(X_1 \leq x_1, X_2 \leq x_2) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} p(u_1, u_2) du_2 du_1$$

where $p(u_1, u_2)$ is the joint PDF. Marginal PDF is obtained by integrating over other variables.

Conditional Probability

$$F(x_1|x_2) = P(X_1 \leq x_1 | X_2 = x_2) = \frac{\int_{-\infty}^{x_1} p(u_1, x_2) du_1}{p(x_2)}$$

$$p(x_1|x_2) = \frac{p(x_1, x_2)}{p_(x_2)}$$

Straightforward extension to multidimensional random variables.

Statistical Independence

CDF and PDF factor into product of marginal probabilities.

$$F(x_1, ..., x_n) = F(x_1)F(x_2)...F(x_n)$$

$$p(x_1, ..., x_n) = p(x_1)p(x_2)...p(x_n)$$

D. Functions of RVs: $Y = g(X)$. Real roots of $y = g(x)$ are $x_1, ...x_n$. Then, PDF of random variable Y is:

$$p_Y(y) = \sum_{i=1}^{n} \frac{p_X(x_i)}{|g'(x_i)|}$$

3.3 Statistics of Random Variables

A. Moments: Let $Y = g(X)$. The nth moment of $Y = g(X)$ is

$$E(Y^n) = \int_{-\infty}^{\infty} [g(x)]^n p(x) dx$$

- If $n = 1$ and $Y = X$, then mean of X, $E(X) \equiv m_x = \int_{-\infty}^{\infty} x p(x) dx$
- If $n = 2$ and $Y = X - m_x$, then variance of X, $\sigma_x^2 = \int_{-\infty}^{\infty} (x - m_x)^2 p(x) dx$

B. Joint Moments:

- Correlation:

$$E(X_i X_j) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_i x_j p(x_i, x_j) dx_i dx_j$$

- Covariance:

$$\mu_{ij} = E[(X_i - m_i)(X_j - m_j)] = E(X_i X_j) - m_i m_j$$

If $E(X_i X_j) = 0$, then RVs X_i and X_j are **orthogonal.**

If $E(X_i X_j) = m_i m_j$, then RVs X_i and X_j are **uncorrelated.**

Statistical independence implies uncorrelation; however, uncorrelation does not necessarily imply independence.

C. Characteristic Function: Defined as a statistical average:

$$\psi_X(jw) = E(e^{jwX}) = \int_{-\infty}^{\infty} e^{jwx} p(x) dx$$

By inverse Fourier transform,

$$p(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi_X(jw) e^{-jwx} dw$$

The moments can be determined from the characteristic function. Since,

$$E(X^n) = (-j)^n \frac{d^n \psi_X(jw)}{dw^n} |_{w=0}$$

They are useful in determining PDF of a sum of statistically independent RVs. Let $Y = \sum_{i=1}^{n} X_i$, then it can be shown that,

$$\psi_Y(jw) = \prod_{i=1}^{n} \psi_{X_i}(jw)$$

If X_i's are independent identically distributed (iid), then $\psi_Y(jw) = [\psi_X(jw)]^n$

3.4 Useful Probability Distributions

A. Gaussian (normal): The PDF is, $p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-m_x)^2/2\sigma^2}$. The CDF can be written as,

$$F(x) = 1 - \text{erfc}\left(\frac{x - m_x}{\sqrt{2}\sigma}\right)$$

where $\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$. Another function often used in $Q(x) = \frac{1}{2}\text{erfc}\ (\frac{x}{\sqrt{2}})$.

The sum of n statistically independent Gaussian RVs (X_i's) is also a Gaussian RV. Let $Y = \sum_{i=1}^{n} X_i$. Then, $m_y = \sum_{i=1}^{n} m_x, \sigma_y^2 = \sum_{i=1}^{n} \sigma_i^2$.

B. Lognormal: Let $X = \ln Y$, where X is normally distributed with mean m and variance σ^2.

$$p(y) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma y} e^{(\ln y - m)/2\sigma^2}, & y \geq 0, \\ 0, & \text{otherwise} \end{cases}$$

Used in modelling the shadowing effect in mobile radio communications.

4 Random Processes

Some random phenomena evolve over time. Examples are:

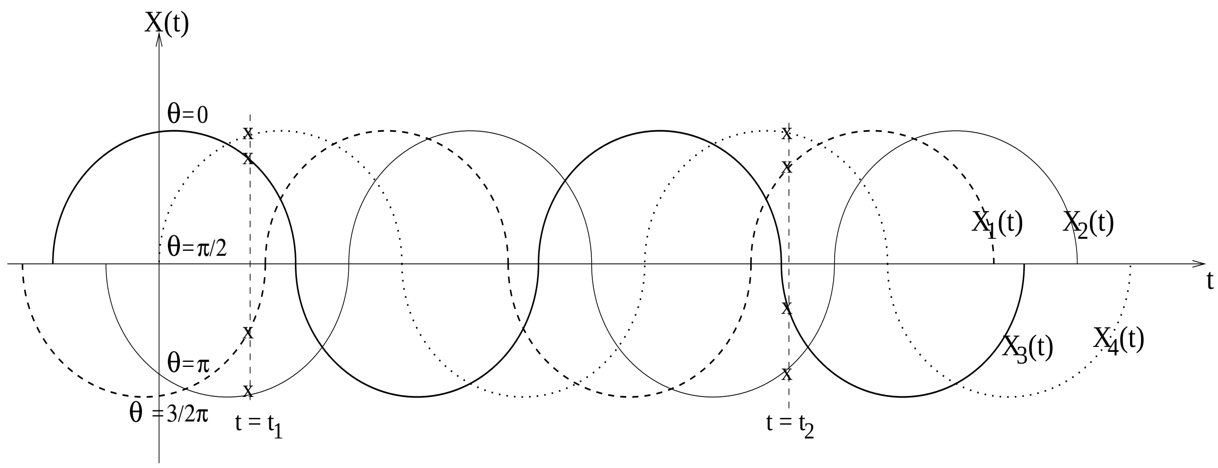
- thermal noise voltages at radio receivers
- audio signals generated during phone conversations

4.1 Notion of Random Process

- A random (or stochastic) process is simply a mapping from the sample space S to a set of functions defined on T, where T is the parametric space (e.g., time).

  ♠ Example: cos $(wt + \theta)$, where $0 \leq \theta \leq 2\pi$.

- A *sample function* is a realization of the random process.
- An *ensemble* (or collection) of sample functions forms a random process.



4.2 Stationary Random Process

Let $X(t)$ be a random process. Then, $X(t_i), (i = 1, 2, ..., n)$ are random variables and can be characterized by their joint PDF, $p(x_{t_1}, ..., x_{t_n})$.

If

$$p(x_{t_1}, ..., x_{t_n}) = p(x_{t_1+t}, ..., x_{t_n+t}), \forall t, n$$

then $X(t)$ is said to be strict sense stationary (SSS). That is, the process is statistically indifferent with respect to any time shift.

4.3 Statistical Averages

Ensemble average: Let $X(t)$ be a random process and $X(t_i) \equiv X_{t_i}$. The nth moment of the RV X_{t_i} is:

$$E(X_{t_i}^n) = \int_{-\infty}^{\infty} x_{t_i}^n p(x_{t_i}) dx_{t_i}$$

If the process is SSS, PDF as well as the moment (hence, $m_{X_{t_i}}$)is independent of time.

Autocorrelation: Consider 2 RVs, $X(t_i), i = 1, 2$.

$$E(X_{t_1} X_{t_2}) \equiv \phi(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_{t_1} x_{t_2} p(x_{t_1}, x_{t_2}) dx_{t_1} dx_{t_2}$$

If $X(t)$ is SS stationary, $p(x_{t_1}, x_{t_2}) = p(x_{t_1+t}, x_{t_2+t})$. As a result, the autocorrelation function is independent of the time instants, t_1 and t_2. Rather, depends on the time difference $(t_1 - t_2)$. That is,

$$E(X_{t_1} X_{t_2}) = \phi(t_1, t_2) = \phi(t_1 - t_2) = \phi(\tau).$$

Note that $\phi(\tau) = \phi(-\tau)$. Important to note that,

$$\phi(0) = E(X_t^2)$$

is the **average power** of the process $X(t)$.

Autocovariance:

$$\mu(t_1, t_2) = E\{[X_{t_1} - m(t_1)][X_{t_2} - m(t_2)]\} = \phi(t_1, t_2) - m(t_1)m(t_2)$$

where $m(t_i), i = 1, 2$, is the mean of X_{t_i}. If $X(t)$ is SS stationary,

$$\mu(t_1, t_2) = \mu(t_1 - t_2) = \mu(\tau) = \phi(\tau) - m^2$$

4.4 Wide Sense Stationary:

Wide sense stationary (WSS) is less stringent than strict-sense stationary (SSS) in that **mean and autocorrelation of the process are constant and dependent on time difference respectively**, but that is non-stationary process (i.e., statistics such as PDF may not be stationary).

Stationary Gaussian Process: Let $X(t)$ be Gaussian process. Then RVs $X(t_i), i = 1, 2, ..., n$, at time instants t_i are jointly Gaussian with mean $m(t_i) = m$ and autocovariance $\mu(t_i, t_j) = \mu(t_i - t_j)$. Gaussian random process is completely characterized by mean and autocovariance functions. If Gaussian process is WSS, it is SSS too since the PDF depends on mean and autocovariance.

4.5 Power Spectrum Density

- Frequency contents of any signal can reveal many features of a signal.
- Note the difference between deterministic signal vs random signal (e.g., cos (ωt) vs cos $(\omega t + \theta)$).
- Sample function of a random process is a random signal.

- Fourier transform of the (deterministic) signal and Fourier transform of the autocorrelation of the (random) signal are respectively used in frequency-domain analysis.

The spectral characteristics (i.e., distribution of power at various frequency components) of a stationary random process is computed by taking the Fourier transform of the autocorrelation function. That is,

$$\boxed{\Psi(f) = \int_{-\infty}^{\infty} \phi(\tau) e^{-j2\pi f\tau} d\tau}$$

$$\boxed{\phi(\tau) = \int_{-\infty}^{\infty} \Psi(f) e^{j2\pi f\tau} df}$$

The average power of the random signal is, $\phi(0) = \int_{-\infty}^{\infty} \Psi(f) df = E(|X_t^2|)$. Whether random process is real or complex, $\Psi(f)$ is real, that is, it contains real information about the actual frequency contents of the (base-band or pass-band) signal.

4.6 Input/Output Relationship in LTI Systems

Let $x(t)$ be the sample function of the stationary random process $X(t)$ and $h(t)$ be the impulse response of a LTI filter. Note that $X(t)$ may be a complex process in general. The output $y(t)$ is a sample function of a process $Y(t)$. Hence,

$$y(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau$$

By convolution (a linear operation),

$$m_y = E[Y(t)] = \int_{-\infty}^{\infty} h(\tau) E[X(t-\tau)] d\tau = m_x \int_{-\infty}^{\infty} h(\tau) d\tau = m_x H(0)$$

where $H(0)$ is the frequency response of the system at $f = 0$. Therefore, the mean of the output process is constant.

The autocorrelation function of the output signal,

$$\phi_{yy}(t_1, t_2) \equiv \frac{1}{2} E(Y_{t_1}, Y^*_{t_2})$$

$$\phi_{yy}(t_1, t_2) = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\beta) h^*(\alpha) E[X(t_1 - \beta) X^*(t_2 - \alpha)] d\alpha d\beta$$

Since $X(t)$ is stationary so is $Y(t)$. Therefore,

$$\phi_{yy}(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\beta) h^*(\alpha) \phi_{xx}(\tau + \alpha - \beta) d\alpha d\beta$$

If a WSS random process is passed through a LTI filter, then the output process is also a WSS process.

It can be shown that the power spectrum density of the output is given by

$$\boxed{\Psi_{yy}(f) = \Psi_{xx}(f) |H(f)|^2}$$

$$\text{Hence, } \phi_{yy}(\tau) = \int_{-\infty}^{\infty} \Psi_{yy}(f) e^{j2\pi f \tau} df = \int_{-\infty}^{\infty} \Psi_{xx}(f) |H(f)|^2 e^{j2\pi f \tau} df$$

The average power of the output signal is

$$\boxed{\phi_{yy}(0) = \int_{-\infty}^{\infty} \Psi_{xx}(f) |H(f)|^2 df}$$

♠ Example From text book, 2.2.1

Note the filtering of the (wideband) white noise results in (narrowband) noise and the finite average power at the filter output.

4.7 Fourier Transform and Properties:

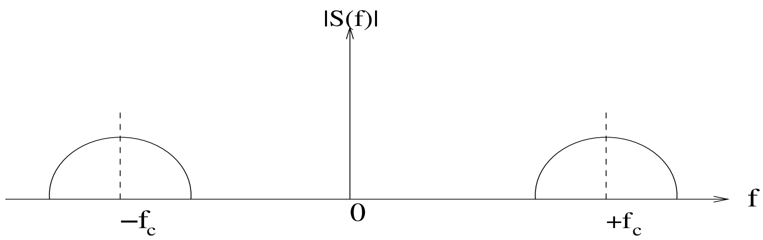
See the table.

5 Complex Signals and Systems

- Many digital signals are transmitted over a channel using some kind of carrier modulation (e.g., DSB). Modulation is to generate bandpass (or passband) signals from baseband signals.
- Representation of different forms of digitally modulated signals are important to understand their spectral characteristics and hence efficient and reliable transmission through the communication channels.
- We want to reduce all bandpass signals/systems to lowpass equivalent signals/systems for easy and consistent analysis. In doing so, bandpass signals/systems are represented as *complex signals/systems.*

♠ Time/Frequency domain description of lowpass and bandpass signals.

5.1 Bandpass Signals



Note that

(a) $e^{j\theta} = \cos\theta + j\sin\theta$ and $e^{-j\theta} = \cos\theta - j\sin\theta$

(b) $\mathrm{Re}(\zeta) = \frac{1}{2}(\zeta + \zeta^*)$

(c) $\cos^2(\theta) + \sin^2(\theta) = 1$

(d) $\cos^2(\theta) - \sin^2(\theta) = \cos(2\theta) \qquad \sin(2\theta) = 2\ \sin\theta\cos\theta$

(e) $\cos(A + B) = \cos A \cos B - \sin A \sin B$

(f) $\sin(A + B) = \sin A \cos B + \cos A \sin B$

Let $s(t)$ be a real-valued bandpass signal and f_c be the carrier frequency. Also let $s_+(t) = s(t) + j\hat{s}(t)$, where $\hat{s}(t)$ is the Hilbert transform of $s(t)$. Therefore,

$$S_+(f) = 2u(f)S(f)$$

Hilbert transformer (or filter) is a 90^o phase-shifter for all frequencies in the input signal. It is given as,

$$h(t) = \frac{1}{\pi t} \qquad \text{and} \qquad H(f) = \begin{cases} -j, & f > 0, \\ +j, & f < 0. \end{cases}$$

♠ Explanation of Hilbert transform using cos $(w_o t)$.

The *analytical signal* $s_+(t)$ is a (complex) bandpass signal. By frequency-shifting, we can obtain an equivalent lowpass signal. Let $S_l(f) = S_+(f + f_c)$. Hence, the equivalent time-domain signal is

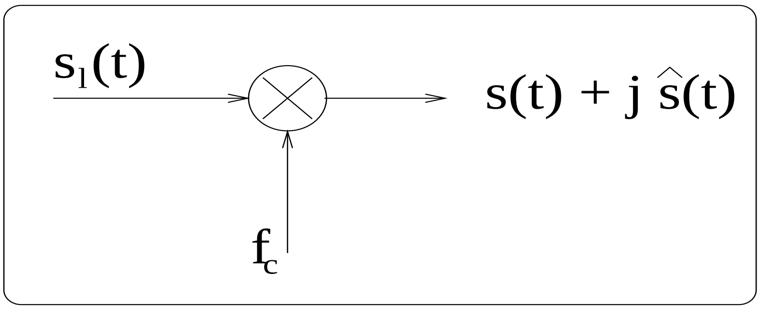
$$s_l(t) = s_+(t)e^{-j2\pi f_c t} = [s(t) + j\hat{s}(t)]e^{-j2\pi f_c t}$$

$$\Rightarrow s(t) + j\hat{s}(t) = s_l(t)e^{j2\pi f_c t}$$

In general, the equivalent lowpass signal $s_l(t)$ is complex-valued and hence,

$$s_l(t) = x(t) + jy(t),$$

where $x(t)$ and $y(t)$ are lowpass real signals. Note that $s(t)$ is bandpass real signal and $s_l(t)$ is lowpass equivalent of bandpass signal $s(t)$.



Quadrature Components: By equating real and imaginary parts in $s_+(t)$ we get,

$$s(t) = x(t)\cos 2\pi f_c t - y(t)\sin 2\pi f_c t$$

$$\hat{s}(t) = x(t)\sin 2\pi f_c t + y(t)\cos 2\pi f_c t$$

where $x(t)$ and $y(t)$ are quadrature components of $s(t)$.

Envelope: Physical meaning of envelope? Example: $m(t)\cos(2\pi f_c t)$, where $m(t)$ is a baseband signal.

Complex Envelope

$$s(t) = \text{Re}[s_l(t)e^{j2\pi f_c t}]$$

$s_l(t)$ is called complex envelope of real signal $s(t)$.

Real Envelope

$$s(t) = a(t)\cos[2\pi f_c t + \theta(t)]$$

where $a(t) = \sqrt{x^2(t) + y^2(t)}$ is the envelope and $\theta(t) = \tan^{-1}\left(\frac{-y(t)}{x(t)}\right)$ is the phase of the real signal $s(t)$.

Frequency Response: Frequency response of the bandpass signal, $s(t)$, and equivalent lowpass signal, $s_l(t)$, are related via,

$$S(f) = \int_{-\infty}^{+\infty} s(t)e^{-j2\pi f t}dt = \frac{1}{2}\int_{-\infty}^{+\infty} [s_l(t)e^{j2\pi f_c t} + s_l^* e^{-2\pi f_c t}]e^{-j2\pi f t}dt$$

$$\boxed{S(f) = \frac{1}{2}[S_l(f - f_c) + S_l^*(-f - f_c)]}$$

Note that $S_l^*(f) = \int_{-\infty}^{+\infty} s_l^*(t) e^{j2\pi ft} dt$.

The energy in the <u>narrowband</u> bandpass signal is,

$$\varepsilon_{\mathrm{S}} = \frac{1}{2}\int_{-\infty}^{\infty} s^2(t)dt = \frac{1}{2}\int_{-\infty}^{\infty} |s_l(t)|^2 dt$$

5.2 Bandpass Systems

A real filter $h(t)$ has:

$$\boxed{H(f) = H_l(f - f_c) + H_l^*(-f - f_c)}$$

where $H^*(-f) = H(f)$ and

$$H_l(f - f_c) = \begin{cases} H(f), & f > 0 \\ 0, & f < 0. \end{cases}$$

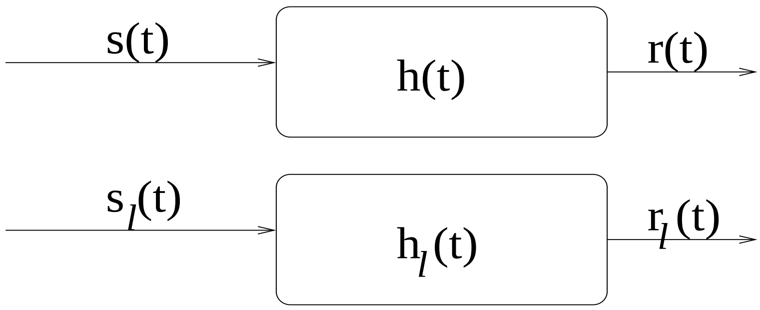
Signal $h(t)$ is given by,

$$h(t) = 2\mathrm{Re}[h_l(t)e^{j2\pi f_c t}]$$

where the equivalent lowpass channel (or system or filter), $h_l(t)$, is in general, complex-valued.

5.3 Response of a Bandpass System to a Bandpass Signal

Let $s(t)$ be bandpass (input) signal and $s_l(t)$ is the equivalent lowpass signal. Also, let $h(t)$ be bandpass (system) response and $h_l(t)$ be equivalent lowpass response.



We have just seen how $s(t)$ and $s_l(t)$ (or $h(t)$ and $h_l(t)$ are related in time and frequency domain. The output of the filter is given by

$$r(t) = \text{Re}[r_l(t)e^{j2\pi f_c t}]$$

where $r(t) = s(t) \star h(t)$.

It can be shown that

$$\boxed{r_l(t) = s_l(t) \star h_l(t)}$$

Also ,

$$\boxed{R(f) = S(f)H(f) \qquad \text{and} \qquad R_l(f) = S_l(f)H_l(f)}$$

$$\boxed{R(f) = \frac{1}{2}[R_l(f - f_c) + R_l^*(-f - f_c)]}$$

Note: The above equivalent characterization of signals and systems are valid for narrowband signals and systems only. Otherwise, the energy in both signals/systems will not be equal.

5.4 Bandpass Random Process

Consider a stationary bandpass process. Similar to signals, relationships between *autocorrelation function* and *power spectrum density* of bandpass and equivalent lowpass process can be derived.

Let $n(t)$ be a sample function of a narrowband WSS process with zero mean and PSD of $\Phi_{nn}(f)$.

Autocorrelation

$$n(t) = a(t)\cos[2\pi f_c t + \theta(t)]$$

$$= x(t)\cos 2\pi f_c t - y(t)\sin 2\pi f_c t$$

$$= \mathrm{Re}[z(t)e^{j2\pi f_c t}],$$

where $a(t) = \sqrt{x^2(t) + y^2(t)}$. Since $n(t)$ is stationary,

$$\phi_{xx}(\tau) = \phi_{yy}(\tau) \qquad \text{and} \qquad \phi_{xy}(\tau) = -\phi_{yx}(\tau)$$

Note that $z(t)$ is low pass equivalent of $n(t)$. It can be shown that

$$\phi_{nn}(\tau) = \phi_{xx}(\tau)\cos 2\pi f_c \tau - \phi_{yx}(\tau)\sin 2\pi f_c \tau$$

Note the relationship between the autocorrelation function of the bandpass process and the autocorrelation and cross-correlation functions of the quadrature components of the process.

Power Spectrum

The lowpass equivalent process, $z(t) = x(t) + jy(t)$. Therefore,

$$\phi_{zz}(\tau) = \frac{1}{2}E[z^*(t)z(t+\tau)] = \frac{1}{2}[\phi_{xx}(\tau) + \phi_{yy}(\tau) - j\phi_{xy}(\tau) + j\phi_{yx}(\tau)]$$

$$\phi_{zz}(\tau) = \phi_{xx}(\tau) + j\phi_{yx}(\tau)$$

Hence,

$$\phi_{nn}(\tau) = \mathrm{Re}[\phi_{zz}(\tau)e^{j2\pi f_c \tau}]$$

That is, autocorrelation function of the bandpass random process is uniquely determined from the autocorrelation of the equivalent lowpass process.

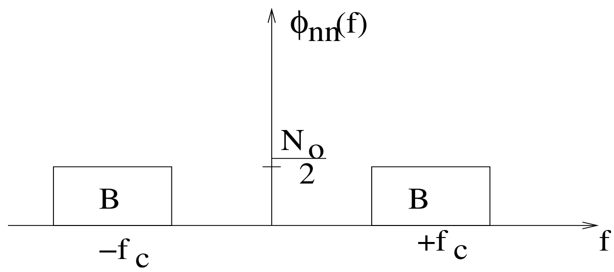
The power spectrum density is given by,

$$\Phi_{nn}(f) = \int_{-\infty}^{\infty} \{\mathrm{Re}[\phi_{zz}(\tau)e^{j2\pi f_c \tau}]\}e^{-j2\pi f \tau} d\tau$$

$$\boxed{\Phi_{nn}(f) = \frac{1}{2}[\Phi_{zz}(f - f_c) + \Phi_{zz}(-f - f_c)]}$$

Note that the autocorrelation function, $\phi_{zz}(\tau)$, is an even function.

5.5 Bandpass White Noise Process



- White noise has a constant power spectrum density over the entire frequency range, i.e., wideband noise signal. Hence, it cannot be expressed in terms of quadrature components.

- Bandpass noise with constant spectrum density is achieved by passing the white noise through an ideal bandpass filter.

- Now that we have a bandpass signal, we can use any of the above three equivalent lowpass random process representation to analyze it.

6 Vector and Signal Space

6.1 Vector Space

A vector $\mathbf{v} = \{v_1 ... v_n\}$ in terms of its basis vectors, $\mathbf{e}_i, 1 \leq i \leq n$,

$$\mathbf{v} = \sum_{i=1}^{n} v_i \mathbf{e}_i$$

v_i is the projection of $\mathbf{v}$ onto the unit vector $\mathbf{e}_i$.

<u>Inner Product:</u> Let $\mathbf{v}_1 = \{v_{11} ... v_{1n}\}$ and $\mathbf{v}_2 = \{v_{21} ... v_{2n}\}$.

$$\mathbf{v}_1 . \mathbf{v}_2 = \sum_{i=1}^{n} v_{1i} v_{2i}$$

The inner product can also be defined as

$$(\mathbf{v}_1 . \mathbf{v}_2) = \parallel \mathbf{v}_1 \parallel \parallel \mathbf{v}_2 \parallel \cos\theta$$

where θ is the angle between the vectors.

$$\parallel \mathbf{v}_1 + \mathbf{v}_2 \parallel^2 = \parallel \mathbf{v}_1 \parallel^2 + \parallel \mathbf{v}_2 \parallel^2 + 2\mathbf{v}_1 . \mathbf{v}_2$$

If $\mathbf{v}_1 . \mathbf{v}_2 = 0, \mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal. Example is $(1,0,0), (0,2,0), (0,0,1)$ in 3-D vector space.

<u>Norm (or Length)</u> of $\mathbf{v}$ is denoted by $\parallel \mathbf{v} \parallel$ and defined by

$$\parallel \mathbf{v} \parallel = (\mathbf{v}.\mathbf{v})^{1/2} = \sqrt{\sum_{i=1}^{n} v_i^2}$$

<u>Orthonormal:</u> If two vectors are orthogonal and each has a unity norm, they are orthonormal.

Gram-Schmidt Procedure for Orthonormalization of Vectors

Given a set of n-dimensional vectors, construct a set of orthonormal vectors.

1. Select any vector ($\mathbf{v}_1$) and normalize it, $\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}$
2. Select the next vector ($\mathbf{v}_2$), subtract the projection of $\mathbf{v}_2$ onto $\mathbf{u}_1$. Thus, $\mathbf{u}_2^{'} = \mathbf{v}_2 - (\mathbf{v}_2.\mathbf{u}_1)\mathbf{u}_1$ and then normalize $\mathbf{u}_2^{'}$. Hence, $\mathbf{u}_2 = \frac{\mathbf{u}_2^{'}}{\|\mathbf{u}_2^{'}\|}$
3. Repeat until n orthonormal vectors are constructed.

♠ Orthonormalize $(0, 0, 2), (0, 1, 1), (1, 0, 0)$

6.2 Signal Space

Let $x_1(t)$ and $x_2(t)$ be (generally complex) signals. The inner product,

$$(x_1(t), x_2(t)) \equiv \int_{-\infty}^{\infty} x_1(t)x_2^*(t)dt$$

- Orthogonal if $(x_1(t), x_2(t)) = 0$
- The norm is defined as

$$\|x(t)\| = \left(\int_{-\infty}^{+\infty} |x(t)|^2 dt\right)^{1/2}$$

- A signal set is linearly independent if no signal can be represented in terms of other signals.

Orthogonal Expansion of Signals

Let $s(t)$ be a real-valued signal with finite energy,

$$\epsilon_s = \int_{-\infty}^{\infty} |s(t)|^2 dt$$

Also let $f_n(t)$ be a set of K orthonormal functions. That is,

$$\int_{-\infty}^{\infty} f_n(t) f_m(t) dt = \begin{cases} 0, & m \neq n, \\ 1, & m = n. \end{cases}$$

$s(t)$ is approximated by $\bar{s}(t) = \sum_{k=1}^{K} s_k f_k(t)$, where $\{s_k, 1 \leq k \leq K\}$ are coefficients in the approximation. Hence, the error in the approximation is

$$e(t) = s(t) - \bar{s}(t)$$

Find $\{s_k\}$ so as to minimize the energy of the error signal ϵ_e.

$$\epsilon_e = \int_{-\infty}^{\infty} \left[s(t) - \sum_{k=1}^{K} s_k f_k(t) \right]^2 dt$$

By mean-square-error (MSE) criterion, minimum ϵ_e is obtained when the error is orthogonal to each of the (basis) functions in the series expansion. Therefore,

$$\int_{-\infty}^{\infty} \left[s(t) - \sum_{k=1}^{K} s_k f_k(t) \right] f_n(t) dt = 0$$

$$\Rightarrow s_n = \int_{-\infty}^{\infty} s(t) f_n(t) dt, \qquad n = 1, ..., K$$

$\bar{s}(t)$ is obtained by projecting the signal $s(t)$ onto K-dimensional signal space.

When $\epsilon_e = 0$, error signal has no energy and the set $f_n(t), (n = 1, ..., K)$ is a complete set. Then,

$$\epsilon_s = \int_{-\infty}^{\infty} |s(t)|^2 dt = \sum_{k=1}^{K} s_k^2$$

♠ Find four orthogonal signals, $\cos(2\pi t)$, $\sin(2\pi t), ...$

Gram-Schmidt Procedure for Orthonormalization of Signals

Let $\{s_i(t), i = 1, ..., M\}$ be a set of finite energy signal waveforms. In general, the orthonormalization of the kth function gives us,

$$f_k(t) = \frac{f_k^{'}(t)}{\sqrt{\epsilon_k}}$$

where

$$f_k^{'}(t) = s_k(t) - \sum_{i=1}^{k-1} c_{ik} f_i(t)$$

and

$$c_{ik} = \int_{-\infty}^{\infty} s_k(t) f_i(t) dt, \qquad i = 1, ..., k-1$$

Once orthonormal signal set is identified, then any signal in the signal space can be expressed in terms of a linear combination of the orthonormal signals.

♠ How to develop basis functions (or signals)?

$$s_1(t) = \begin{cases} 1, & 0 < t < 2, \\ 0, & \text{otherwise.} \end{cases} \quad s_2(t) = \begin{cases} 1, & 0 < t < 1, \\ -1, & 1 < t < 2, \\ 0, & \text{otherwise.} \end{cases}$$

$$s_3(t) = \begin{cases} 1, & 0 < t < 2, \\ -1, & 2 < t < 3, \\ 0, & \text{otherwise.} \end{cases} \quad s_4(t) = \begin{cases} -1, & 0 < t < 3, \\ 0, & \text{otherwise.} \end{cases}$$

1. $f_1(t) = \sqrt{\frac{1}{2}} s_1(t)$

2. $c_{12} = 0$, therefore, $f_2(t) = \sqrt{\frac{1}{2}} s_2(t)$

3. $c_{13} = \sqrt{2}$ and $c_{23} = 0$, hence,

$$f_3^{'}(t) = s_3(t) - \sqrt{2} f_1(t) = \begin{cases} -1, & (2 \leq t \leq 3), \\ 0, & \text{otherwise.} \end{cases}$$

$f_3(t) = f_3^{'}(t)$

4. $f_4^{'}(t) = 0$, eliminate this signal from the orthogonal set.

In signal form,

$$s_4(t) = -\sqrt{2} f_1(t) + f_3(t)$$

and in vector form,

$$s_4(t) = (-\sqrt{2}, 0, 1)$$

7 Source Coding

- Analog sources have **infinite** number of output values whereas digital sources have **finite** number of output values. Sampling (in x-axis) and quantization (in y-axis) produces digital signals from analog signals.

- Source encoder generates efficient digital sequence (i.e., binary: 0's and 1's) for transmission/storage.

7.1 Information Source Models

- Source output is random and hence is characterized by statistical terms.
- Binary source emits a binary sequence of the form 1010001001101 ... where the output alphabet set consists of two letters $\{0, 1\}$, with transmitted waveforms, for example, $x_0(t) = \cos\ (2\pi t)$ and $x_1(t) = \sin\ (2\pi t), 0 \leq t \leq T$.

- In general, a digital information source emits a sequence of letters (or symbols) selected from a *finite alphabet set* of L possible letters (or symbols), $\{x_1, ..., x_L\}$.

$$\spadesuit L = 16, \text{ with 4 bits/symbol.}$$

Let p_k be the occurrence probability of a symbol x_k, $\quad (1 \le k \le L)$. Hence,

$$p_k = P(X = x_k), \qquad \text{and} \qquad \sum_{k=1}^{L} p_k = 1$$

- Source output sequence may be *statistically independent* (e.g., pixels in a frame of a video clip with space).

  They are known as Discrete Memoryless Sources (DMS) for which joint PDFs can be expressed as the product of (individual) marginal PDFs.

- Source output sequence may be *statistically dependent* (e.g., English text).

  We can have a mathematical model based on (statistically) stationary property for which joint probabilities of any output sequence are invariant with respect to time shift.

7.2 Nyquist Sampling Theorem

Let $X(t)$ be a stationary random process with power spectral density $\Phi_{XX}(f)$ and $\Phi_{XX}(f) = 0$ for $|f| \ge W$ (i.e., band-limited). By sampling theorem,

$$X(t) = \sum_{n=-\infty}^{\infty} X\left(\frac{n}{2W}\right) \frac{\sin\,[2\pi W(t - \frac{n}{2W})]}{[2\pi W(t - \frac{n}{2W})]}$$

where discrete samples $X(m), m = 1, 2, ...$ are taken at the Nyquist sampling rate of $2W$ samples/s.

$$\spadesuit \text{ Bandwidth, sinc}(x), \text{ construction of } X(t) \text{ from } X\left(\frac{n}{2W}\right).$$

It is important to note the following:

- Analog output, $X(t) \Rightarrow$ equivalent discrete-time output, $X(m)$.
- Output samples are generally continuous and cannot be represented in digital form without loss in precision (lossy quantization process).

7.3 Measure of Information

Let x_i and y_j $(i = 1, ..., n,$ and $j = 1, ..., m)$ be two random variables. Let

$$P(X = x_i | Y = y_j) \equiv P(x_i | y_j).$$

What if X and Y are independent (e.g., raining in Toronto and NASDAQ index going up or down) or dependent (e.g., wearing winter jackets in a snowy day)?

If X and Y are independent, then

$$P(x_i|y_j) = P(x_i), \qquad \text{no information is carried by } y_i \text{ about } x_i.$$

If X and Y are totally dependent $(P(x_i) = P(y_j))$, then

$$P(x_i|y_j) = \frac{P(y_j)}{P(y_j)} = 1, \qquad \text{full information is carried by } y_i \text{ about } x_i.$$

Hence, $P(x_i|y_j)$ carries some information!

7.3.1 Mutual Information

Mutual information between random variables x_i and y_j is defined by,

$$\boxed{I(x_i; y_j) = \log \left[\frac{P(x_i|y_j)}{P(x_i)}\right]}$$

From the above definition, statistically independent events carry no information, i.e., $I(.) = 0$. Totally dependent events carry $I(x_i; y_j) = \log \frac{1}{P(x_i)} = -\log P(x_i)$ amount of information. Note that $0 \leq P(x_i) \leq 1$. Also it can be seen that, $I(X;Y) = I(Y;X)$.

Self-Information: of the event $X = x_i$,

$$\boxed{I(x_i) = -\log P(x_i)}$$

Highly (low) probable event carries less (more) information.

Additivity Property: Let a discrete information source emit binary digits (0 or 1) every T seconds with equal probability. Hence, the information measure (content) is

$$I(x_i) = -\log_2 P(x_i), \qquad x_i \in \{0, 1\}$$

$$= -\log_2 \frac{1}{2} = 1 \text{ bit} \qquad \text{(note the base of 2)}$$

Let us now consider a block of K binary digits that are statistically independent (DMS model). There are $M = 2^K$ possible K-bit blocks, each with equal occurrence probability of 2^{-K}. Note that

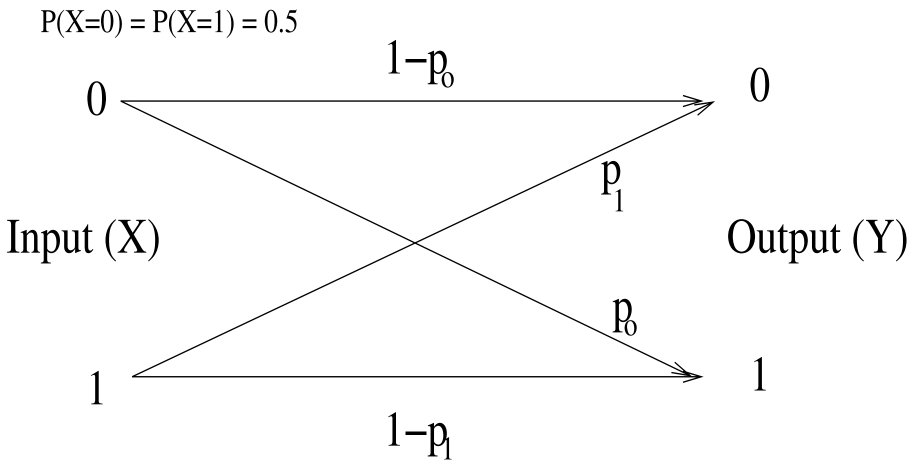
$$p(x_1...x_N) = p(x_1)...p(x_N)$$

Hence the self-information of the block is

$$I(x_1...x_K) = -\log_2(2^{-K}) = K \text{ bits.}$$

Hence, the logarithmic measure of information possess the additive property for independent discrete sources.

Information Carried by Binary Input Binary Output (BIBO) Channel



Given that $Y = 0$, what is the mutual information about the occurrence of $X = 0$?

$$I(x;y) = \log_2 \frac{P(x|y)}{P(x)}$$

Since $I(x;y) = I(y;x)$,

$$I(X=0;Y=0) = I(Y=0;X=0) = \log_2 \frac{P(Y=0|X=0)}{P(Y=0)}$$

Also,

$$P(Y=0) = P(Y=0|X=0)P(X=0) + P(Y=0|X=1)P(X=1),$$

$$\Rightarrow P(Y=0) = \frac{1}{2}(1 - p_o + p_1)$$

$$\text{Hence, } I(0;0) = \log_2 \left[\frac{2(1-p_o)}{1-p_o+p_1} \right]$$

If $p_o = p_1 = 0$ (noiseless channel), then $I(0,0) = 1$.

If $p_o = p_1 = \frac{1}{2}$, then $I(0,0) = 0$ (useless channel).

7.3.2 Entropy

Average mutual information is defined as,

$$I(X;Y) = \sum_{i=1}^{n}\sum_{j=1}^{m} P(x_i, y_j) I(x_i; y_j) = \sum_{i=1}^{n}\sum_{j=1}^{m} P(x_i, y_j) \log \left[\frac{P(x_i, y_j)}{P(x_i)P(y_j)}\right]$$

If X and Y are independent, $I(X;Y) = 0$.

The entropy of X, $H(X)$, is defined as average *self-information* per letter (or symbol) where X represents all possible output letters (or symbols) from a source. That is,

$$H(X) = \sum_{i=1}^{n} P(x_i) I(x_i) = -\sum_{i=1}^{n} P(x_i) \log\, P(x_i).$$

If $P(x_i) = 1/n, \forall i$, then $H(X) = \log\, n$. If $n = 16, H(X) = 4$ bits/symbol.

It can be shown that, $H(X) \leq \log\, n$. That is, **the entropy of a discrete source is maximum when the output symbols are equally probable**.

Let us consider a BSC where $p_o = p_1 = p$. Also let $P(X = 0) = q$ and $P(X = 1) = 1 - q$. Hence,

$$H(X) = -q\log\, (q) - (1 - q)\log\, (1 - q).$$

From the figure, $H(X)$ is maximum when $q = \frac{1}{2}$.

Conditional entropy is defined as

$$H(X|Y) = \sum_{i=1}^{n}\sum_{j=1}^{m} P(x_i, y_j) I(x_i|y_j) = -\sum_{i=1}^{n}\sum_{j=1}^{m} P(x_i, y_j) \log\, P(x_i|y_j)$$

- $H(X|Y)$ can be interpreted as the average amount of uncertainty (conditional self-information) in X after we observe Y.

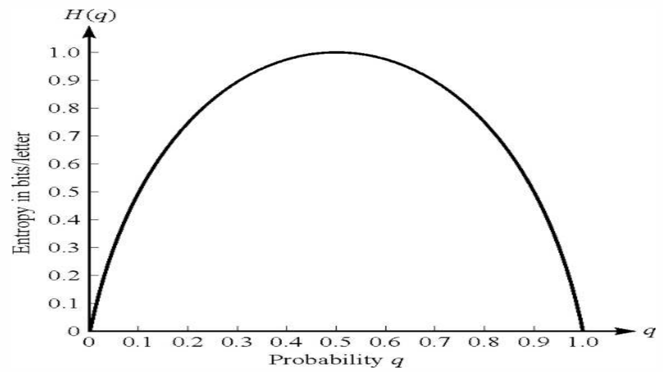


Figure 3.1−1
Binary entropy function.

- Uncertain events have large information (surprises!) and certain events (i.e., highly predictable) have small information (no excitement!). Hence, entropy of events with near-zero probability have high information.

- $H(X)$ can be interpreted as average amount of uncertainty (self-information) prior to the observation

- It can be shown that the mutual information, $I(X;Y)$,

$$\boxed{I(X;Y) = H(X) - H(X|Y)}$$

  Therefore, $I(X;Y)$ is the average amount of uncertainty (mutual information) provided by X by the observation of Y.

- Note that conditioning decreases the uncertainty.

7.4 Coding for Discrete Sources

- $H(X)$ measures the average information in the discrete source output.

- Encoding process: representing the source output efficiently by a sequence of binary digits (i.e., compression).

- Coding efficiency : compare the average number of binary digits per output symbol with source entropy, $H(X)$. It is defined as $H(X)/\bar{R}$ where $\bar{R}$ is the average number of bits required to represent a symbol.

Let us consider a DMS source that produces an output symbol x_i , $i = 1, ..., L$, every T seconds. Each symbol occurs at the source output with probability, $P(x_i)$. The average self information (i.e., entropy) is,

$$H(X) \leq \log_2 L.$$

Source rate is $H(X)/T$ bits/s.

7.4.1 Fixed Length Coding

Consider a block encoding scheme in which a *unique* set of R binary digits (i.e., code words) are assigned to each symbol. Total number of symbols is L (assume L is a power of 2). Hence, $R = \log\, _2 L$. Hence, the average number of bits/symbol,

$$\bar{R} = \sum_{i=1}^{L} R \times P(x_i) = R.$$

Since, $H(X) \leq \log_2 L$, $R \geq H(X)$. In fixed-length coding scheme with equiprobable occurrences, the coding efficiency is $\frac{H(X)}{\bar{R}} = \frac{H(X)}{R} = 1$

However, when L is not a power of 2, then $R = \lfloor \log_2 L \rfloor + 1$ and the coding efficiency is $\leq \frac{\log_2 L}{\lfloor \log_2 L \rfloor + 1}$. Then, if $L \gg 1$, the efficiency is high and if L is small, efficiency is low.

Shannon Source Coding Theorem:

Let X be an alphabet set (consisting of output symbols) with entropy $H(X)$. Block of J symbols are encoded into code words of length N digits from a binary alphabet. The symbols are *not uniquely* encoded and therefore decoding error occurs.

The probability of a block decoding failure (P_b) can be made very small if

$$\hat{R} = \frac{N}{J} \geq H(X)$$

and J is sufficiently large.

In other words, the average number of bits per symbol required to encode the output of a DMS with near-zero decoding error probability is lower bounded by its entropy (i.e., average information content).

Let $X = \{x_0, x_1, x_2\}$ and it is given that $H(X) = 3/4$. If $J = 3, N = 2$, i.e., a block of 3 symbols $(x_0x_1x_2, x_0x_2x_1, x_1x_0x_2)$ is encoded with unique code word $(11, 01, 10)$ respectively. The rest of the possible blocks are encoded to one code word (00). In this example, $J = 3, N = 2$ and $H(X) = 3/4 > \frac{2}{3} = 2/3$. Therefore, decoding failure probability will approach to 1. On the other hand, if $J = 3, N = 3$, then $H(X) = 3/4 < \frac{3}{3} = 1$ and the decoding failure can be made smaller and, in fact to zero because of unique coding.

7.4.2 Variable Length Coding

When source symbols are not equally probable, use variable-length code words. Idea is to assign short code words for high probable source symbols (known as *entropy coding*).

♠ Example: Variable Length Coding

Symbol	$P(a_k)$	Code I	Code II	Code III
a_1	$\frac{1}{2}$	1	0	0
a_2	$\frac{1}{4}$	00	10	01
a_3	$\frac{1}{8}$	01	110	011
a_4	$\frac{1}{8}$	10	111	111

Code I has problem in decoding. What is the decoded sequence if the received sequence is 001001 ? $a_2a_1a_2a_1$ or $a_2a_4a_3$?

The prefix condition requires that no code word is a prefix of any other code word. Code III is also not uniquely decodable.

Huffman Coding Algorithm

(a) Variable-length code words, (b) prefix condition will be satisfied, (c) average number of binary digits to represent the source symbols is minimum, (d) uniquely and instantaneously decodable, (e) symbol probabilities are needed and (f) code words may not be unique but all achieve the same efficiency.

♠ Example using Huffman Coding

The average number of bits/symbol, $\bar{R} = \sum_{k=1}^{7} n_k P(x_k)$, where x_k is the symbol and n_k is number of bits in the code word to represent x_k.

It can be seen that $H(X) = 2.11$ and $\bar{R} = 2.21$ in the above example (efficiency = 95%).

Symbol	Probability	Code word
x_1	0.35	00
x_2	0.30	01
x_3	0.20	10
x_4	0.10	110
x_5	0.04	1110
x_6	0.005	11110
x_7	0.005	11111

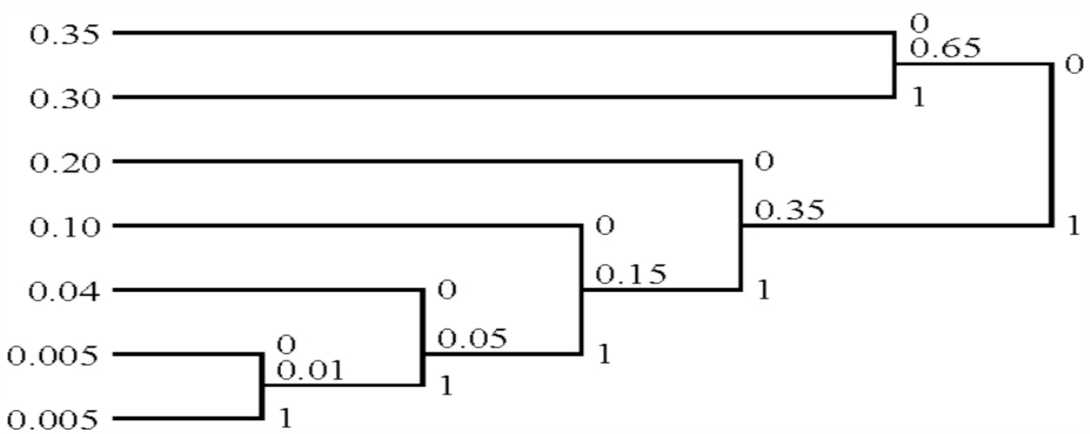


Figure 3.3–4
An example of variable–length source encoding for a DMS.

Instead of encoding symbol-by-symbol, if we do the encoding blocks of symbols (e.g., pair), $\bar{R}$ can be decreased (see example 3.3.3).

7.5 Coding for Analog Sources

- When $X(t)$ is a band-limited stationary random process, the sampling theorem allows us to represent $X(t)$ by a sequence of uniform samples taken at the Nyquist rate.

- Samples are then quantized in amplitude and encoded with finite number of bits. Output is grouped into L levels with R bits/sample where $R = \log_2 L$. Example, in PCM (used in speech coding), $L = 4096, R = 12$.

- Entropy (Huffman) coding can be used to efficiently represent (i.e., compress) the discrete sequences. However, distortion is also introduced, because continuous levels are converted into discrete levels.

Rate Distortion Function: Let x_k and $\hat{x}_k$ be actual and quantized values. Squared-error distortion is: $d(x_k, \hat{x}_k) = (x_k - \hat{x}_k)^2$. With n samples,

$$d(\mathbf{X}_n, \hat{\mathbf{X}}_n) = \frac{1}{n} \sum_{k=1}^{n} d(x_k, \hat{x}_k)$$

Expected value of $d(\mathbf{X}_n, \hat{\mathbf{X}}_n)$ is defined as **distortion** D. Therefore,

$$D = E[d(\mathbf{X}_n, \hat{\mathbf{X}}_n)] = \frac{1}{n} \sum_{k=1}^{n} E[d(x_k, \hat{x}_k)] = E[d(x, \hat{x})]$$

Rate distortion function $R(D)$ is defined as the **minimum number of bits per source output** that is required to represent the output X of the memoryless source with distortion D.

Rate Distortion Function for Memoryless Gaussian Source

$$R_g(D) = \begin{cases} \frac{1}{2}\log_2(\sigma_x^2/D), & 0 \le D \le \sigma_x^2 \\ 0, & D > \sigma_x^2. \end{cases}$$

where σ_x^2 is the variance of the Gaussian source output. Distortion-rate function (for the above Gaussian source) is, $D_g(R) = 2^{-2R}\sigma_x^2$. In dB, $-6R + 10\log_{10}\sigma_x^2$. Note that mean-square error distortion decreases at a rate of 6dB/bit. What happens when $D > \sigma_x^2$?

8 Channel Capacity

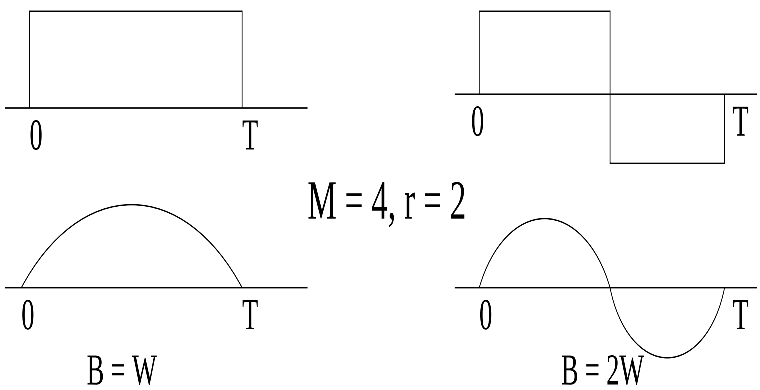
8.1 Channel Coding

Introduce redundancy in the binary information sequence at the **channel encoder** that can be used at the receiver to overcome the effects of noise and interference encountered in the physical channel.

$$k \text{ raw bits } \rightarrow n \text{ coded bits (or code word)}$$

Code rate is denoted as (k,n). The raw rate becomes $\frac{n}{k}$ with the amount of redundancy $= \left(\frac{n-k}{k}\right)\%$. Example: $a_1a_2 \rightarrow a_1a_2a_1 \oplus a_2$.

- Sequence of coded bits are input into the **modulator** which transmits one symbol for each r bits (using one waveform from $M = 2^r$ possible waveforms). Orthogonal signals can make P_e smaller but with larger bandwidth

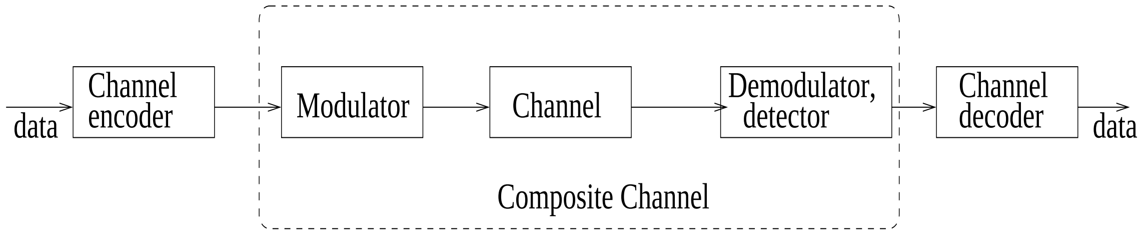


  expansion factor, $B_e = \frac{W}{R}$, where R is the channel bit rate. B_e grows exponentially with block length, r.

- **Demodulator** produces an estimate of the transmitted data symbols (binary or M-ary / scalar or vector) that have been possibly corrupted.

- **Detector** outputs 0's and 1's (based on soft/hard decision rule).
- **Channel decoder** exploits the redundancy available in the code word to compensate for the channel impairments.

8.2 Channel Models



Discrete-Input Discrete-Output Channels

BIBO and BSC are examples. In general, we can have $q-$ary inputs and Q-ary outputs. $q = Q = 2$ gives BIBO channel.

Discrete-Input Continuous-Output Channels

Let a channel encoder output consists of symbols selected from a finite alphabet set, $X = \{x_0, ..., x_{q-1}\}$ and $Q = \infty$ (un-quantized input to the channel decoder). The composite channel can be characterized by the conditional PDFs as ,

$$P(y|X = x_k), \qquad k = 0, 1, ..., q - 1.$$

Let us consider a AWGN channel. Then,

$$Y = X + G$$

where G is a zero-mean Gaussian RV with variance σ^2. For a given $X = x_k$, Y is Gaussian with mean x_k and variance σ^2. Therefore,

$$P(y|X = x_k) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(y-x_k)^2/2\sigma^2}$$

If we transmit n symbols through the memoryless channel, we can say,

$$P(y_1, ..., y_n|X = x_1, ..., X_n = x_n) = \prod_{i=1}^{n} P(y_i|X_i = x_i)$$

Continuous-Input Continuous-Output Channels

Let us suppose that the inputs to the channel and outputs from the channel are continuous. Let $x(t)$ be a band-limited (to W) input process to a AWGN channel (with PSD of $\frac{1}{2}N_0$). Let $n(t)$ be the sample function of the noise process. Hence, the output process,

$$y(t) = x(t) + n(t)$$

By expanding the signals with orthornormal functions,

$$x(t) = \sum_i x_i f_i(t) \qquad n(t) = \sum_i n_i f_i(t) \qquad y(t) = \sum_i y_i f_i(t)$$

where $\{x_i\}, \{n_i\}, \{y_i\}$ are the sets of coefficients in the corresponding expansion. The functions $\{f_i(t)\}$ form a complete orthonormal set over $(0, T)$, i.e.,

$$\int_0^T f_i(t) f_j(t) dt = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

- Any orthonormal functions can be used in the expansion of white Gaussian $n(t)$

- Note that n_i's are uncorrelated and hence statistically independent Gaussian RVs. Hence, y_i's are also statistically independent Gaussian RVs. Hence,

$$y_i = \int_0^T y(t) f_i(t) dt = \int_0^T [x(t) + n(t)] f_i(t) dt = x_i + n_i$$

- Now we can work with the coefficients in characterizing the channel. Hence

$$P(y_i|x_i) = \frac{1}{\sqrt{2\pi}\sigma_i} e^{-(y_i - x_i)^2/2\sigma_i^2}, \qquad i = 1, 2, ...$$

- Continuous waveform channel has been reduced to equivalent discrete channel.

- Samples of $x(t)$ and $y(t)$ can be taken at Nyquist rate of $2W$ samples/s. Hence, in an interval of T, a total of $2WT$ samples will be taken. Hence, $2WT$ transition probabilities are enough to describe the AWGN channel.

- The variance of y_i is, $\sigma_i^2 = \frac{1}{2} N_0$. $\sigma_i^2 = \mathrm{E}[y_i^2]$ = Average power contained in sample y_i.

8.3 Capacity of BSC and AWGN Channels

Consider a discrete memoryless channel having an input alphabet $X = \{x_0, ..., x_{q-1}\}$ and an output alphabet $Y = \{y_0, ..., y_{Q-1}\}$ with the set of transition probabilities, $P(y_i|x_i)$.

The mutual information provided about $X = x_i$ by the occurrence of $Y = y_i$ is,

$$I(x_i; y_i) = \log \left[\frac{P(y_i|x_i)}{P(y_i)} \right]$$

The probability of $Y = y_i$ is,

$$P(y_i) = \sum_{k=0}^{q-1} P(x_k) P(y_i|x_k).$$

Therefore, the average mutual information provided by the output Y about the input X is given by,

$$I(X;Y) = \sum_{j=0}^{q-1} \sum_{i=0}^{Q-1} P(x_j)P(y_i|x_j)\log\left[\frac{P(y_i|x_j)}{P(y_i)}\right]$$

- Channel characteristics determine $P(y_i|x_j)$
- Probabilities of input symbols, $P(x_j), \forall j$, depend on the channel encoder
- Goal is to maximize the average mutual information, $I(X;Y)$. If we do it over all $P(x_j)$'s, then the maximized quantity depends only on channel-dependent $P(y_i|x_j)$.

The channel capacity is defined as,

$$\boxed{C = \max_{P(x_j)} I(X;Y)}$$

under the constraints that $P(x_j) \geq 0$ and $\sum_{j=0}^{q-1} P(x_j) = 1$. The units of C is bits/symbol and the channel rate is C/T, if a symbol enters the channel every T seconds.

♠ Capacity of a BSC

Given that $P(0|1) = P(1|0) = p$ and $P(0) = P(1) = 0.5$. The capacity of a BSC is

$$C = p\text{log } 2p + (1-p)\text{log } 2(1-p) = 1 - H(p)$$

where $H(p)$ is the source entropy.

♠ AWGN Channel: Discrete Input and Continuous Output

Transition probabilities are given as,

$$P(y|X = x_k) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(y-x_k)^2/2\sigma^2}$$

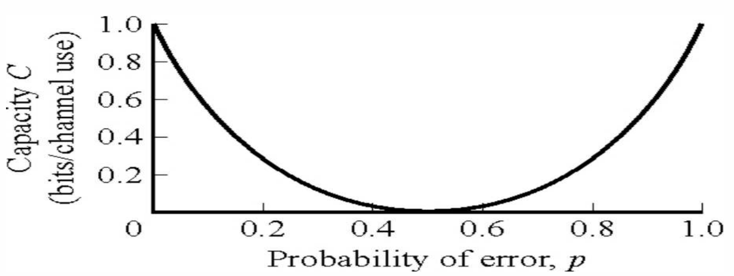


Figure 7.1–4
The capacity of a BSC as a function of the error probability p.

where $X = \{x_0, ..., x_{q-1}\}$ and $Y = \{-\infty, +\infty\}$. Therefore,

$$C = \max_{P(x_i)} \sum_{i=0}^{q-1} \int_{-\infty}^{\infty} P(y|x_i) P(x_i) \log_2 \frac{P(y|x_i)}{P(y)} dy$$

where

$$P(y) = \sum_{k=0}^{q-1} P(y|x_k) P(x_k)$$

For example, let $X = \{A, -A\}$ and $P(A) = P(-A) = 0.5$. $I(X;Y)$ is maximized when X's are equi-probable.

Therefore, the capacity of binary-input AWGN memoryless channel is,

$$C = \frac{1}{2} \int_{\infty}^{\infty} P(y|A) \log_2 \left[\frac{P(y|A)}{P(y)} \right] dy + \frac{1}{2} \int_{\infty}^{\infty} P(y|-A) \log_2 \left[\frac{P(y|-A)}{P(y)} \right] dy$$

Note that not always do the equi-probable input symbols nor the symmetric transition probabilities maximize the average mutual information (or the channel capacity).

8.4 Shannon's Channel Capacity

Let us consider a band-limited channel with AWGN. The Shannon's capacity is defined as,

$$C = \lim_{T \to \infty} \max_{p(x)} \frac{1}{T} I(X;Y)$$

Let us use the samples (or coefficients) to determine $I(X;Y)$. We want to find the mutual information between $X_N = [x_1 ... x_N]$ and $Y_N = [y_1 ... y_N]$, where $N = 2WT$ and $y_i = x_i + n_i$.

The capacity of the band-limited AWGN channel with a band-limited and average power-limited input is shown to be,

$$\boxed{C = W \log_2\left(1 + \frac{P_{av}}{WN_0}\right) = W \log_2(1 + \text{SNR})}$$

where P_{av} is the average power of $x(t)$. The relationship between SNR and C

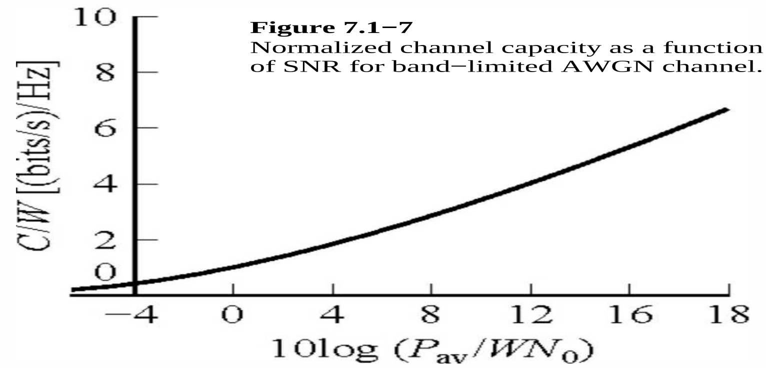


Figure 7.1−7
Normalized channel capacity as a function of SNR for band−limited AWGN channel.

is given in the figure.

- For a fixed SNR, C is proportional to W.
- If W is fixed, C increases with P_{av} in a channel.
- If P_{av} is fixed, C can be increased by increasing W in a channel.

$$P_{av} = C\varepsilon_b, \qquad \varepsilon_b \text{ is energy per bit.}$$
$$\frac{\varepsilon_b}{N_0} = \frac{2^{C/W} - 1}{C/W}$$

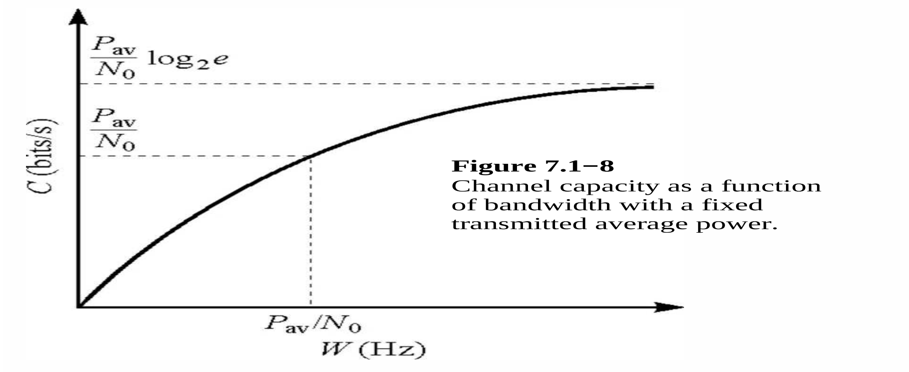


Figure 7.1–8
Channel capacity as a function of bandwidth with a fixed transmitted average power.

$$\lim_{\frac{C}{W}\to 0}\left(\frac{\varepsilon_b}{N_0}\right) \rightarrow \ln 2$$

Noisy Channel Coding Theorem: There exist channel codes that can achieve reliable communication with $P_e \rightarrow 0$, if the transmission rate $R < C$.

9 Digitally Modulated Signals

- Modulator is as an interface that maps digital information (i.e., sequence of binary digits) into analog waveforms.
- Random channel characteristics and random inputs have to be considered in the (deterministic) mapping of symbols into waveforms.
- Performance measure: bandwidth of waveforms, signal energy and probability of bit or symbol error.

9.1 Memoryless Modulation

Modulator with memory operates on current and previous digits (or waveforms). The waveforms may differ in:

Amplitude: binary PAM (or ASK); carrier-modulated PAM: DSB, SSB

Phase: BPSK, QPSK; and QAM (PAM-PSK)

Frequency: FSK, MSK

9.1.1 Amplitude Shift Keying (ASK)

ASK signals are also known as Pulse-Amplitude-Modulated (PAM) signals.

$$s_m(t) = A_m g(t) \cos 2\pi f_c t, \qquad m = 1, 2, ..., M, \qquad 0 \le t \le T$$

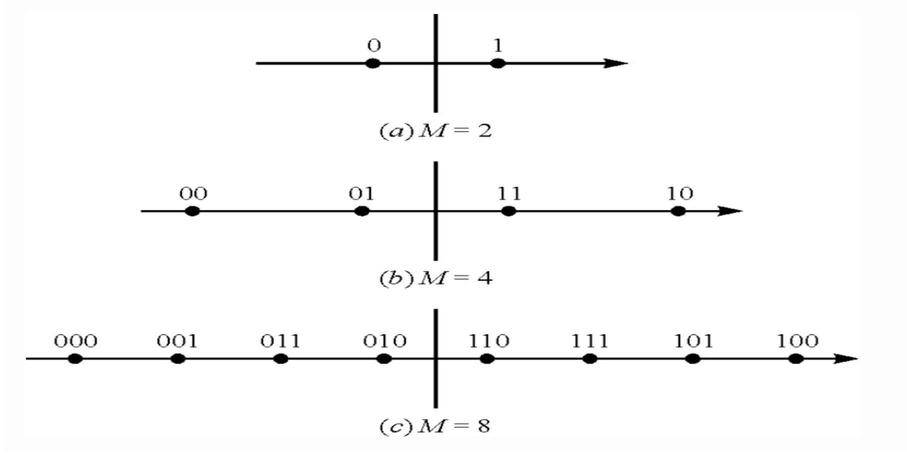
where A_m denotes the set of M possible amplitudes corresponding to $M(= 2^k)$ possible $k-$bit blocks of symbols.

- $A_m = (2m - 1 - M)d$, where $2d$ is the distance between adjacent signal amplitudes. ♠$M = 4 : (-3d, -d, +d, +3d)$
- The waveform $g(t)$ is a real-valued signal pulse whose shape influences the spectrum of the transmitted signal (flexible in spectrum shaping)
- If input rate to the modulator is R bits/s, then the symbol rate is R/k symbols/s

The energy of signal, $s_m(t)$ is

$$\varepsilon_m = \int_0^T s_m^2(t)dt = \frac{A_m^2}{2} \int_0^T g^2(t)dt = \frac{A_m^2}{2}\varepsilon_g$$

where ε_g is the signal energy in the pulse $g(t)$.
Let $s_m(t) = s_m f(t)$ where $f(t)$ is the unit-energy signal. Note that one-dimensional signal set is enough to expand the waveform. Hence, $s_m = A_m\sqrt{\frac{\varepsilon_g}{2}}$.



The Euclidean distance between amplitudes of signals $s_m(t)$ and $s_n(t)$ is,

$$d_{mn}^{(e)} = \sqrt{(s_m - s_n)^2} = \sqrt{\frac{\varepsilon_g}{2}}|A_m - A_n| = d\sqrt{2\varepsilon_g}|m - n|.$$

Therefore, minimum Euclidean distance is

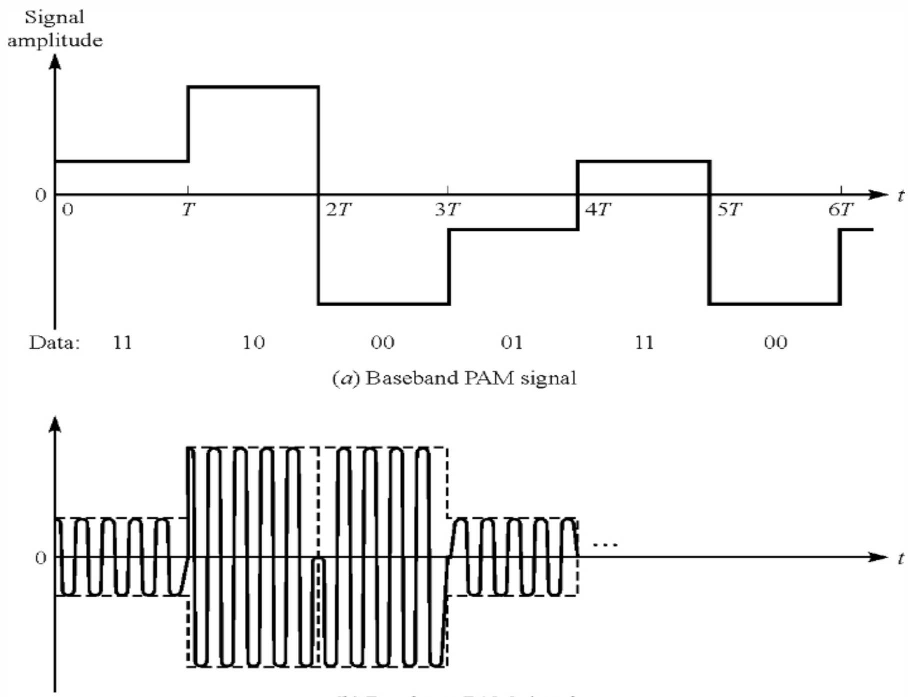
$$d_{min}^{(e)} = d\sqrt{2\varepsilon_g}$$

The signal-space diagrams are shown in the figure for different M. They are also known as **constellations**.

Antipodal Signal: $M = 2, \qquad s_1(t) = -s_2(t), \qquad \varepsilon_1 = \varepsilon_2$

Gray Encoding: Mapping of k information bits to M possible amplitudes is done in such a way that adjacent signal amplitudes differ by one binary digit. In the case of (most likely) errored detection of the transmitted symbol, only a single bit error is occurred.

The digital PAM signal can also be used in baseband transmission (i.e., no modulation). In such a case $s_m(t) = A_m g(t)$. Figure shows the baseband PAM and Passband PAM.



(a) Baseband PAM signal

(b) Bandpass PAM signal

9.1.2 Phase Shift Keying (PSK)

In digital phase modulation, the M signal waveforms are represented as

$$s_m(t) = g(t)\cos\left[2\pi f_c t + \frac{2\pi(m-1)}{M}\right], \quad m = 1, ..., M, \quad 0 \le t \le T.$$

$$s_m(t) = g(t)\cos\theta_m \cos 2\pi f_c t - g(t)\sin\theta_m \sin 2\pi f_c t$$

where $\theta_m = 2\pi(m-1)/M$.

- θ_m's are possible phases of the carrier that convey the transmitted information.
- These PSK signals have equal energy of $\frac{\varepsilon_g}{2}$ for all the transmitted waveforms where ε_g is the signal energy of the pulse $g(t)$.

- These signals are two-dimensional and hence can be represented as a linear combination of two orthonormal signal waveforms, $f_1(t)$ and $f_2(t)$.

Therefore,

$$s_m(t) = s_{m1}f_1(t) + s_{m2}f_2(t)$$

where,

$$f_1(t) = \sqrt{\frac{2}{\varepsilon_g}}g(t)\ \cos\ 2\pi f_c t$$

$$f_2(t) = -\sqrt{\frac{2}{\varepsilon_g}}g(t)\ \sin\ 2\pi f_c t$$

The 2D vector is given by,

$$\mathbf{s}_m = \left[\sqrt{\frac{\varepsilon_g}{2}}\ \cos\ \frac{2\pi(m-1)}{M} \quad \sqrt{\frac{\varepsilon_g}{2}}\ \sin\ \frac{2\pi(m-1)}{M}\right]$$

The minimum Euclidean distance between adjacent phases of the signals is,

$$d_{min}^{(e)} = \sqrt{\varepsilon_g\left[1 -\ \cos\ \left(\frac{2\pi}{M}\right)\right]}$$

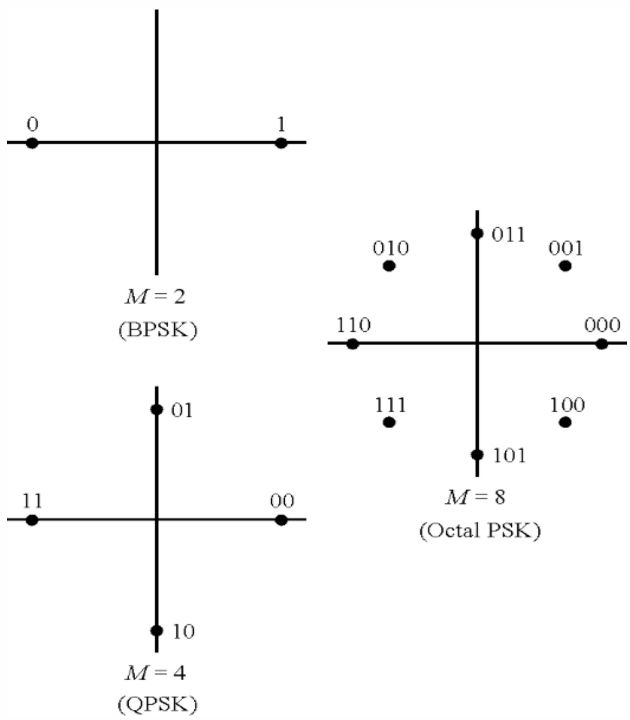
Signal space diagrams are shown in the figure. $M = 2$ (BPSK) corresponds to binary PAM (also antipodal signals). The preferred mapping of symbols to waveforms is using the Gray coding.

♠Constellation and Waveforms for$M = 2, 4, 8$.

9.1.3 Quadrature PAM (QAM)

It can be viewed as the combination of ASK and PSK. In QAM, two symbols are efficiently modulated using two quadrature carriers, $\cos\ (2\pi f_c t)$ and $\sin\ (2\pi f_c t)$. Hence,

$$s_m(t) = A_{mc}g(t)\ \cos\ (2\pi f_c t) - A_{ms}g(t)\ \sin\ 2\pi f_c t.$$



♠Constellation and Waveforms for$M = 8$

Note the $\frac{\pi}{4}$ shift to increase the distance between signal points.

9.1.4 Frequency Shift Keying (FSK)

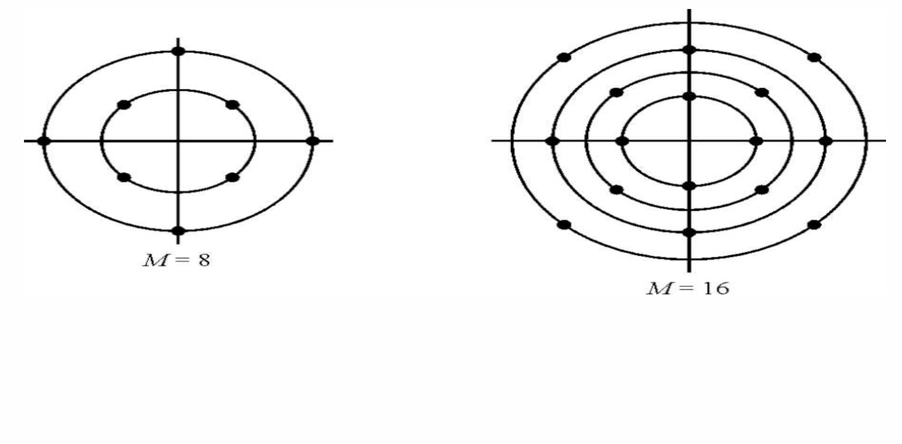
Let us consider M signals that have equal energy of ε with duration of T.

$$s_m(t) = \sqrt{\frac{2\varepsilon}{T}} \cos\left[2\pi(f_c + m\triangle f)t\right], \quad m = 1, ...M, \quad 0 \le t \le T.$$

The cross-correlation coefficient between signals $s_m(t)$ and $s_k(t)$ is

$$\rho_{km} = \frac{\sin\left[2\pi T(m-k)\triangle f\right]}{2\pi T(m-k)\triangle f}.$$

For every $\frac{1}{2T}$, ρ_{km} is zero, i.e., signals are orthogonal. Since $|m - k| = 1$ corresponds to adjacent frequency slots, $\triangle f = \frac{1}{2T}$ represents the minimum



frequency separation between orthogonal signals. M (orthogonal) FSK signals can be represented as vectors

$$\mathbf{s}_1 = [\sqrt{\varepsilon} \quad 0 \quad 0 ... \quad 0 \quad 0]$$

$$\mathbf{s}_M = [0 \quad 0 \quad 0 ... \quad 0 \quad \sqrt{\varepsilon}]$$

- The Euclidean distance between pairs of signal is

$$d_{km}^{(e)} = \sqrt{2\varepsilon}, \quad \forall m, k$$

- FSK signals are generated by shifting the carrier frequency f_c by

$$\triangle f_n = \frac{1}{2} \triangle f I_n, \quad I_n = \pm 1, \pm 3, ..., \pm M - 1.$$

- The frequency shift is at least $\triangle f = \frac{1}{2T}$ for orthogonality.
- Since $\triangle f = \frac{1}{2T}$, the smaller pulse (i.e., high speed communication) requires larger frequency separation, and hence the FSK signal bandwidth is increased.
- ♠$f_c = 10^6 \text{Hz}, T = 5\mu s$

- ♠Constellation for $M = 4, \quad I_n = \pm 1, \pm 3$

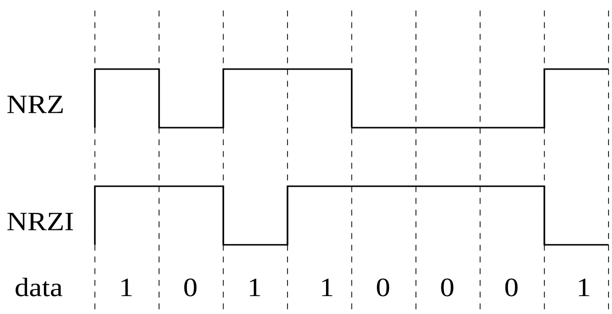
$$\triangle f_1 = \frac{\triangle f}{2}, \triangle f_2 = -\frac{\triangle f}{2}, \triangle f_3 = \frac{3\triangle f}{2}, \triangle f_4 = -\frac{3\triangle f}{2}$$

9.2 Modulation with Memory

- There exists a dependence between the signals transmitted in the successive symbol intervals.
- Dependence is introduced to shape the spectrum of the transmitted signals to match the channel characteristics.
- This is achieved by encoding the data sequence at the modulator input by means of modulation code.

Example with NRZI

Let us consider a modulation scheme where "0" and "1" are represented by a rectangular pulse of $-A$ and A respectively.



- Memory is introduced in this signal by the pre-coding operation.

- Transition occurs only when a "1" is transmitted. This is *differential encoding* and is described mathematically as:

$$b_k = a_k \oplus b_{k-1}$$

where a_k is the binary information sequence into the encoder and b_k is the output sequence of the encoder.

- The output of the encoder ($b_k's$) is mapped into one of two waveforms.

9.2.1 Continuous Phase FSK (CPFSK)

CPFSK is a continuous phase modulation scheme.

- FSK signals are generated by shifting the carrier frequency by

$$f_n = \frac{1}{2}\triangle f I_n, \quad I_n = \pm 1, \pm 3, ..., \pm M - 1.$$

- Within the symbol duration of T, oscillator has to tune to one of M frequencies. This abrupt switching (i.e., $m\Delta f, m = 1, 2..$) results in large spectral side lobes and hence the FSK signal bandwidth is increased.

Solution: information-bearing signal frequency-modulates a single carrier whose frequency is changed continuously. The phase of the carrier is also continuous (CPFSK). It is a modulation scheme with memory.

♠ Rectangle pulse and constant modulation index.

Let $d(t)$ be a PAM signal,

$$d(t) = \sum_n I_n g(t - nT),$$

where $g(t)$ is a rectangle pulse of duration T with amplitude $\frac{1}{2T}$. $\{I_n\}$ is the sequence of amplitudes obtained by mapping k-bit blocks of binary digits from symbols $\{a_n\}$ into different levels, $M = 2^k$. Therefore, the carrier-modulated signal with equal-energy of ε,

$$s(t) = \sqrt{\frac{2\varepsilon}{T}} \cos\left[2\pi f_c t + \phi(t; \mathbf{I})\right],$$

where

$$\phi(t; \mathbf{I}) = 4\pi T f_d \int_{-\infty}^{t} d(\tau) d\tau = 4\pi T f_d \int_{-\infty}^{t} [\sum_n I_n g(\tau - nT)] d\tau$$

where f_d is the maximum frequency deviation.

Note that $d(t)$ may have discontinuities but $\int d(t)$ is continuous.

The phase of the carrier in interval, $nT \leq t \leq (n+1)T$ is given by

$$\phi(t; \mathbf{I}) = \theta_n + 2\pi h I_n q(t - nT), \qquad nT \leq t \leq (n+1)T$$

where,

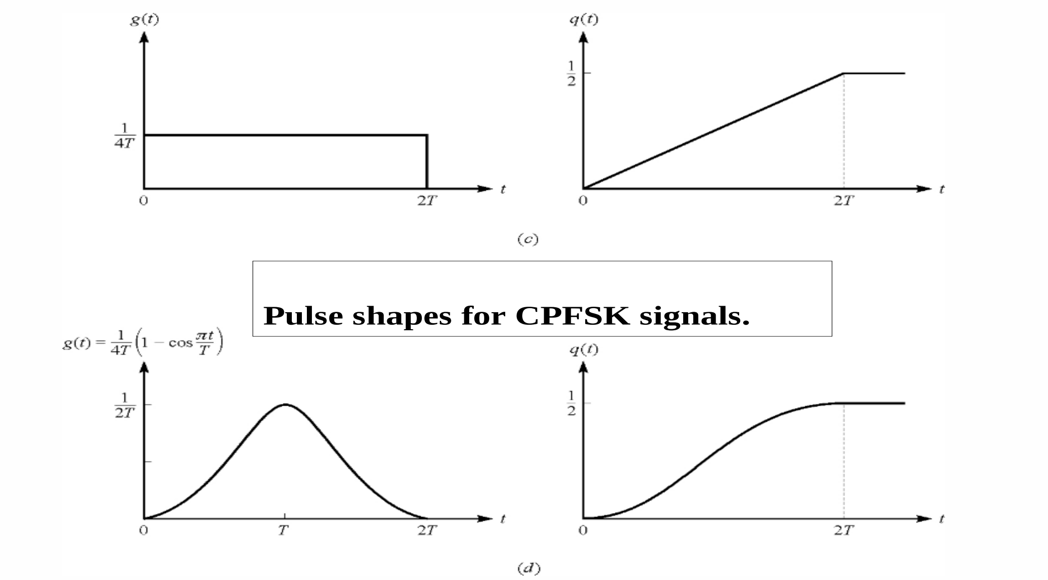
$$h = 2 f_d T$$

$$\theta_n = \pi h \sum_{k=-\infty}^{n-1} I_k$$

$$q(t) = \begin{cases} 0 & t < 0, \\ t/2T & 0 \leq t \leq T, \\ 1/2 & t > T. \end{cases}$$

- Note the accumulation of memory in θ_n.
- h is called modulation index.
- In general,

$$q(t) = \int_0^t g(\tau) d\tau.$$



Phase Trees: Phase of the PSK signals are described using phase-evolution tree.

Let $I_n = \pm 1$ be binary symbols and the corresponding phase tree is shown in the figure. Assume $\theta_0 = 0, g(t)$ is a rectangle pulse with duration T.

Smoother phase trajectories can be obtained using pulses that do not contain discontinuities.

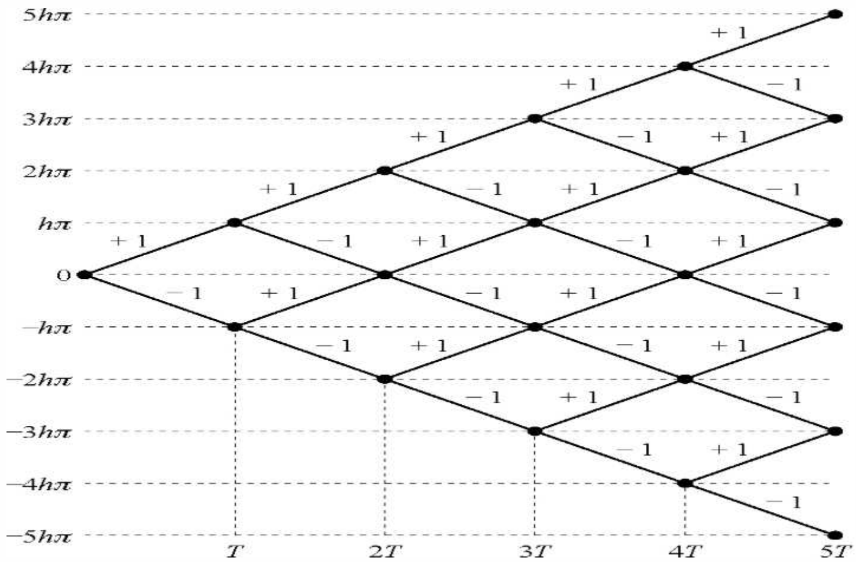
9.2.2 Minimum Shift Keying (MSK):

Binary CPFSK with $h = \frac{1}{2}$. Hence, the maximum frequency deviation, $f_d = \frac{1}{4T}$ and

$$\phi(t; \mathbf{I}) = \theta_n + \pi I_n \left(\frac{t - nT}{2T} \right), \quad nT \leq t \leq (n+1)T$$

Therefore,

$$s(t) = A \cos \left[2\pi f_c t + \theta_n + \frac{\pi}{2} I_n \left(\frac{t - nT}{2T} \right) \right]$$



Phase trajectory for binary CPFSK signal with rectangular pulse.

$$s(t) = A \cos \left[2\pi \left(f_c + \frac{1}{4T} I_n \right) t - \frac{\pi}{2} n I_n + \theta_n \right]$$

The binary ($I_n = \pm 1$) CPFSK signals can be expressed as a sinusoid having one of two possible frequencies in the interval $nT \leq t \leq (n+1)T$. Let

$$f_1 = f_c - \frac{1}{4T} \qquad f_2 = f_c + \frac{1}{4T}$$

Minimum frequency $\triangle f = \frac{1}{2T}$. Hence, binary CPFSK with $h = \frac{1}{2}$ is called MSK.

Comparison of MSK and QPSK signals:

- MSK signals have continuous phase
- QPSK has either $\pm\pi$ or $\pm\pi/2$ phase jumps
- OQPSK has $\pm\pi/2$ phase jumps

See the annexed figure.

10 Spectra of Digital Signals

- Spectral content of the digitally modulated signals is important.
- Information sequence is random and hence the modulated signal is a random process.
- PSD will be used in the design and analysis of signals.

Let us consider a digital linear modulation with no carrier modulation in which,

$$v(t) = \sum_{n=-\infty}^{\infty} I_n g(t - nT),$$

where $\{I_n\}$ is a sequence that is obtained by mapping of symbols $\{a_n\}$. Note that I_n is real in PAM and complex in PSK or QAM.

The autocorrelation function of $v(t)$ is

$$\phi_{vv}(t; t+\tau) = \frac{1}{2}\mathrm{E}[v(t)v^*(t+\tau)] = \frac{1}{2}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} E[I_n^* I_m] g^*(t-T) g(t-mT+\tau).$$

Assume that $\{I_n\}$ is wide-sense stationary with mean μ_i and autocorrelation $\phi_{ii}(m) = \frac{1}{2}E[I_n^* I_{n+m}]$. Then,

$$E[v(t)] = \mu_i \sum_{n=-\infty}^{\infty} g(t - nT).$$

It can also be shown that $\phi_{vv}(\tau)$ is periodic. That is,

$$\phi_{vv}(t + T; t + \tau + T) = \phi_{vv}(t; t + \tau)$$

Hence, $v(t)$ is a cyclo-stationary process and PSD is computed by averaging out within the symbol period T. Therefore,

$$\bar{\phi}_{vv}(\tau) = \frac{1}{T}\int_{-T/2}^{T/2} \phi_{vv}(t; t+\tau)dt$$

The Fourier transform of $\bar{\phi}_{vv}(\tau)$ gives average PSD of $v(t)$ as,

$$\boxed{\Phi_{vv}(f) = \frac{1}{T}|G(f)|^2\Phi_{ii}(f)}$$

where $G(f)$ is the Fourier transform of $g(t)$ and $\Phi_{ii}(f)$ is the PSD of the information sequence which is defined as,

$$\Phi_{ii}(f) = \sum_{m=-\infty}^{\infty} \phi_{ii}(m)e^{-j2\pi fmT}.$$

That is, the spectral characteristics of $v(t)$ can be controlled by design of pulse $g(t)$ as well as correlation properties of the information sequence.

10.1 Spectral Shaping without Precoding

If symbols are mutually uncorrelated, then

$$\phi_{ii}(m) = \begin{cases} \sigma_i^2 + \mu_i^2, & m = 0, \\ \mu_i^2, & \text{otherwise}, \end{cases}$$

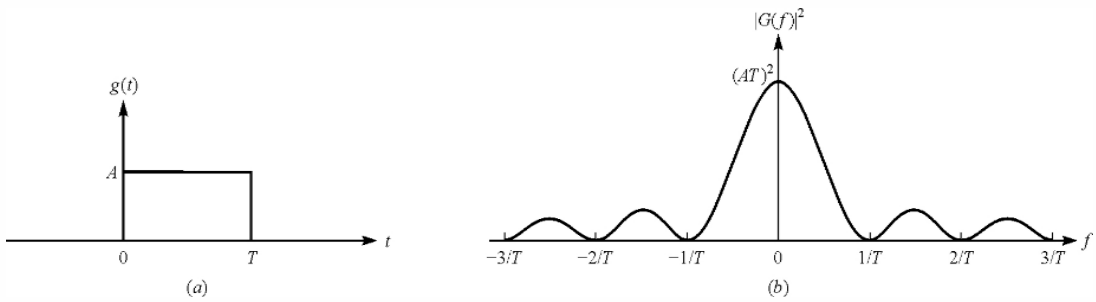
where σ_i^2 denotes the variance of the information sequence $\{I_n\}$.

For uncorrelated real symbols,

$$\boxed{\Phi_{vv}(f) = \frac{\sigma_i^2}{T}|G(f)|^2 + \frac{\mu_i^2}{T^2}\sum_{m=-\infty}^{\infty}|G(\frac{m}{T})|^2\delta(f - m/T)}$$

The first term is continuous spectrum and second term consists of discrete frequency components spaced $1/T$ apart in frequency.

What if $\{I_n\}$ are equi-probable and symmetrically positioned in the signal space? $\mu_i = 0$.



Rectangular pulse and its energy density spectrum

10.1.1 Rectangular pulse

Let $g(t)$ be a rectangular pulse with height A and duration T.

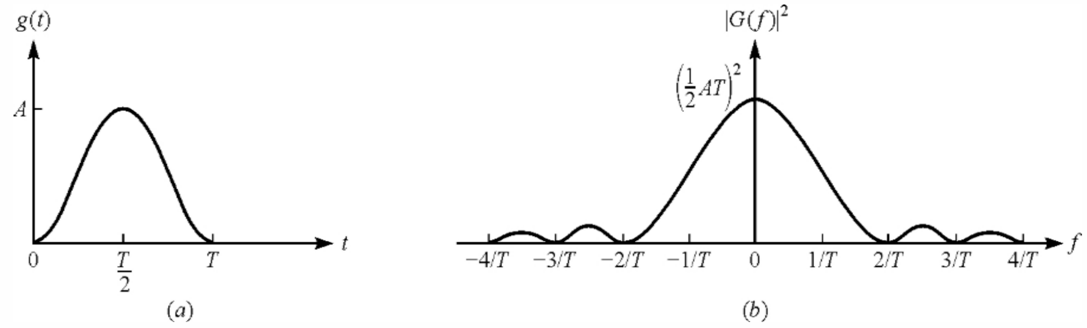
$$G(f) = AT\frac{\sin(\pi fT)}{(\pi fT)}e^{-j\pi fT}$$

This spectrum contains zeros at multiples of $1/T$ in frequency and it decays in proportion to $\frac{1}{f^2}$. Therefore,

$$\Phi_{vv}(f) = \sigma_i^2 A^2 T\left(\frac{\sin \pi fT}{\pi fT}\right)^2 + A^2\mu_i^2\delta(f)$$

10.1.2 Raised cosine pulse

$$g(t) = \frac{A}{2}\left[1 + \cos\frac{2\pi}{T}(t - T/2)\right], \quad 0 \le t \le T.$$
$$G(f) = \frac{AT}{2}\frac{\sin \pi fT}{\pi fT(1 - f^2T^2)}e^{-j\pi fT}$$



Raised cosine pulse and its energy density spectrum

Power spectrum of the raised cosine decays faster but wider than rectangle pulse.

10.2 Spectral Shaping with Precoding

Let

$$I_n = b_n + b_{n-1}$$

Assume $\{b_n\}$'s are uncorrelated random variables each having zero mean and unit variance. Then

$$\phi_{ii}(m) = \begin{cases} 2, & m = 0, \\ 1, & m = \pm 1, \\ 0, & \text{otherwise.} \end{cases}$$

The PSD of the input sequence is

$$\Phi_{ii}(f) = 2(1 + \cos(2\pi f T)) = 4\cos^2(\pi f T)$$

and the corresponding PSD of the (lowpass) modulated signal is

$$\Phi_{vv}(f) = \frac{4}{T}|G(f)|^2 \cos^2(\pi f T)$$

10.3 PSD of Bandpass Signals

In previous sections, we used baseband signals to analyze the frequency contents of the digital signals using PSD. Let $s(t)$ be a band-pass signal,

$$s(t) = \text{Re}[v(t)e^{j2\pi f_c t}],$$

where $v(t)$ is the equivalent low-pass signal of $s(t)$. The autocorrelation function of $s(t)$ is,

$$\phi_{ss}(\tau) = \text{Re}[\phi_{vv}(\tau)e^{j2\pi f_c \tau}].$$

Hence, the PSD is

$$\Phi_{ss}(f) = \frac{1}{2}[\Phi_{vv}(f - f_c) + \Phi^*_{vv}(-f - f_c)].$$

11 Receiver Design for AWGN Channel

When the channel is corrupted by AWGN,

- design of optimum receiver
  - demodulator: correlator, matched filter
  - symbol detector (MAP, ML and MD criterion)
  - sequence detector (ML, Viterbi algorithm)
- performance analysis (P_e)

Consider M-ary transmission with signal waveforms

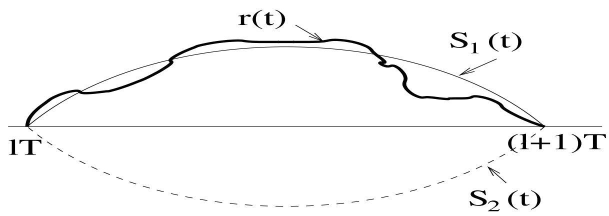
$$s_m(t), \quad 0 \le t \le T, \quad m = 1, 2, ..., M.$$

Let the dimension of the M-ary signal set be N, i.e., the orthonormal functions $f_n(t), \quad n = 1, 2, ..., N$, span the signal space.

Received signal in the symbol interval T is,

$$r(t) = s_m(t) + n(t),$$

where $n(t)$ is the sample function of the AWGN process with PSD of $\Phi_{nn}(f) = \frac{N_0}{2}$.



Binary anti–podal signals:
$S_2(t) = -S_1(t)$

In the figure, intuitively, if $\int r(t)dt > 0$, then we can declare that the transmitted pulse is $s_1(t)$

- Observe $r(t)$ over T and then decide what was transmitted to the best of knowledge. That is, design a receiver that is optimum in the sense that it minimizes the probability of making an error.

Receiver consists of a demodulator and a detector.

Demodulator converts the received signal $r(t)$ into N dimensional vector by projecting $r(t)$ onto N orthonormal functions, $\mathbf{r}= [r_1 \quad r_2 \quad ... \quad r_N]$.

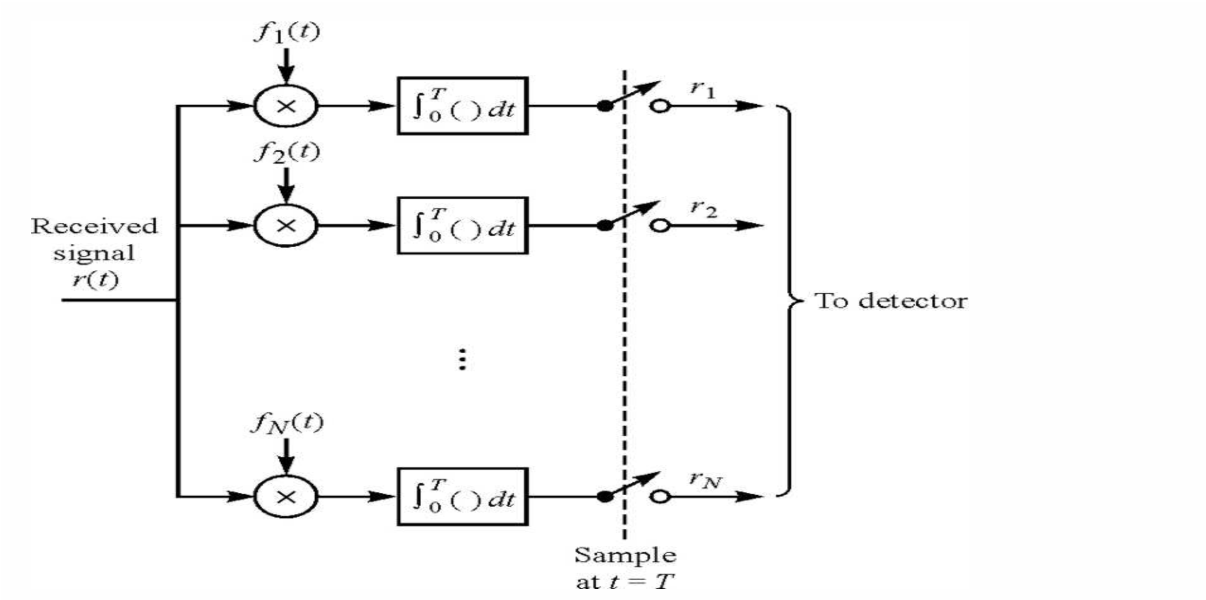
Detector decides which of the M possible waveforms was transmitted based on received vector $\mathbf{r}$.

11.1 Demodulator

Two optimum realizations of the demodulator: correlator and matched filter.

11.1.1 Correlator

- The signal $s_m(t)$ and the noise $n(t)$ are expanded into a series of linearly weighted orthonormal functions.
- Note that $\{f_n(t)\}$ does not span the noise space.
- The received signal $r(t)$ is passed through a parallel bank of N cross correlators that compute the projection of $r(t)$ onto N orthonormal functions $\{f_n(t)\}$ over T, as shown in the figure.



Consider k^{th} branch between time interval lT and $(l+1)T$,

$$\int_{lT}^{(l+1)T} r(t) f_k(t) dt = \int_{lT}^{(l+1)T} [s_m(t) + n(t)] f_k(t) dt$$

$$\Rightarrow r_k = s_{mk} + n_k, \qquad k = 1, 2, ...N,$$

where

$$s_{mk} = \int_{lT}^{(l+1)T} s_m(t) f_k(t) dt \quad \text{and} \quad n_k = \int_{lT}^{(l+1)T} n(t) f_k(t) dt.$$

- Transmitted signal $s_m(t)$ is represented by $\mathbf{s}_m = [s_{m1} \quad ... \quad s_{mN}]$.
- The components $\{n_k\}$ are random variables that arise due to the presence of additive noise.
- The received signal $r(t)$ can now be written as,

$$r(t) = \sum_{k=1}^{N} r_k f_k(t) + n^{'}(t)$$

It can be shown that $n^{'}(t)$ and r_k are statistically independent and consequently, $n^{'}(t)$ is irrelevant to the decision as to what was transmitted. Therefore, the decision can be based on the correlator output (signal and noise) components, $r_k = s_{mk} + n_k$.

Statistics of $\{n_k\}$ and $\{r_k\}$

N noise components $\{n_k\}$ are zero-mean uncorrelated Gaussian RVs with equal variance of $\sigma_n^2 = \frac{1}{2}N_0$. Therefore, the correlator outputs $\{r_k\}$ conditioned on the m^{th} signal being transmitted are Gaussian RVs with mean and variance given by,

$$E(r_k) = E(s_{mk} + n_k) = s_{mk}$$

$$\sigma_r^2 = \sigma_n^2 = \frac{1}{2}N_0$$

The correlator outputs (r_k) are statistically independent Gaussian RVs. Therefore,

$$P(\mathbf{r}|\mathbf{s}_m) = \prod_{k=1}^{N} P(r_k|s_{mk}), \qquad m = 1, 2, ..., M,$$

where,

$$P(r_k|s_{mk}) = \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{(r_k - s_{mk})^2}{N_0}\right]$$

Joint conditional PDFs,

$$\boxed{P(\mathbf{r}|\mathbf{s}_m) = \frac{1}{(\pi N_0)^{N/2}} \exp\left[-\sum_{k=1}^{N} \frac{(r_k - s_{mk})^2}{N_0}\right]}$$

Sufficient Statistics

All the relevant information to make the decision as to which of $s_m(t)$ was transmitted is contained in the correlator outputs $\{r_k\}$ where $r_k = s_{mk} + n_k$.

♠ M-ary baseband PAM signal

M-ary baseband PAM with basic pulse shape $g(t)$. $A_m = (2m - 1 - M)d$. PAM signal has dimension $N = 1$. Let

$$g(t) = \begin{cases} A, & 0 \le t \le T, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, $\varepsilon_g = A^2T$.

$$f(t) = \frac{1}{A\sqrt{T}} g(t) = \begin{cases} 1/\sqrt{T}, & 0 \le t \le T, \\ 0, & \text{otherwise.} \end{cases}$$

The output of the correlator output is,

$$r = \int_0^T r(t)f(t)dt = \frac{1}{\sqrt{T}} \int_0^T r(t)dt = \frac{1}{\sqrt{T}} \int_0^T [s_m(t) + n(t)]dt$$

$$\Rightarrow r = s_m + n.$$

In PAM, $s_m = A_m = (2m - 1 - M)d$. As shown earlier, $E(n) = 0, E(r) = s_m$ and $\sigma_r^2 = \sigma_n^2 = \frac{N_0}{2}$. Hence, the PDF of the sampled output (conditioned on that $s_m(t)$ was transmitted) can be specified as,

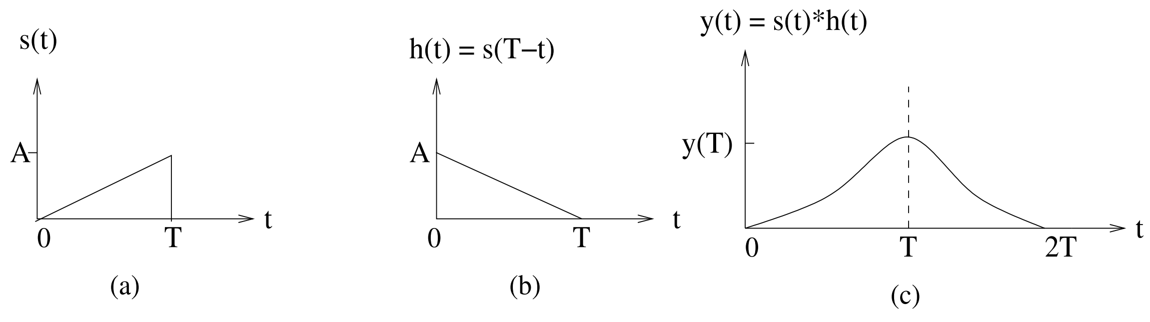
$$P(r|s_m) = \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{(r - s_m)^2}{N_0}\right]$$

♠ What if $M = 2$ for anti-podal signals, $s_1 = -s_2$? What is P_e?

♠ Design of the correlator?

11.1.2 Matched Filter

Instead of N correlators, N linear (matched) filters can be used in the demodulation.



Let

$$h_k(t) = f_k(T - t), \qquad 0 \le t \le T,$$

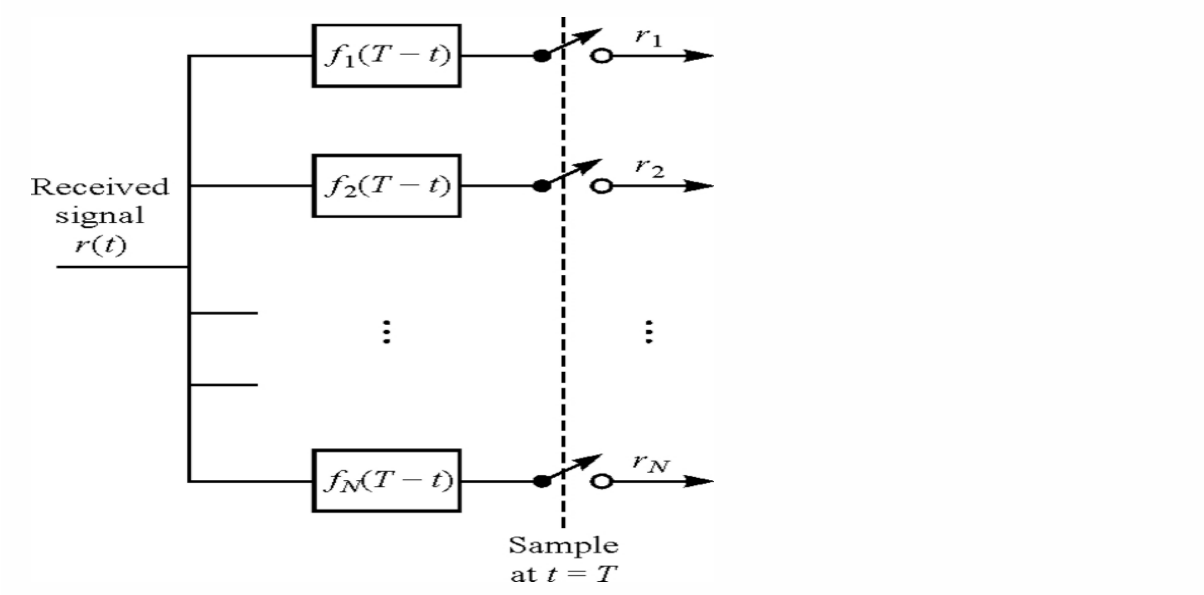
be the impulse response of the N filters where $\{f_k(t)\}$ are the N basis functions and $h_k(t) = 0$ outside the symbol interval T. The output of the filter is,

$$y_k(t) = \int_0^t r(\tau)h_k(t-\tau)d\tau = \int_0^t r(\tau)f_k(T-t+\tau)d\tau, \qquad k = 1, 2, ..., N.$$

If we sample at $t = T$,

$$y_k(T) = \int_0^T r(\tau)f_k(\tau)d\tau = r_k$$

Therefore, the sampled outputs of the filters at $t = T$ are the exact values obtained from N correlators.



Computation of Output SNR

If $s(t)$ is corrupted by AWGN, the filter with an impulse response matched to $s(t)$ **maximizes** the output SNR. [See the text book for the proof, pp. 237-239.]

Let us compute the SNR using frequency-domain approach.

$$H(f) = \int_0^T s(T-t)e^{-j2\pi ft}dt = \left[\int_0^T s(\tau)e^{j2\pi f\tau}d\tau\right]e^{-j2\pi fT}$$

$$H(f) = S^*(f)e^{-j2\pi fT}$$

The output spectrum is,

$$Y(f) = |S(f)|^2 e^{-j2\pi fT}.$$

Hence,

$$y_s(t) = \int_{-\infty}^{\infty} |S(f)|^2 e^{-j2\pi fT} e^{j2\pi ft} df$$

By sampling at $t = T$, we obtain,

$$y_s(T) = \int_{-\infty}^{\infty} |S(f)|^2 df = \int_0^T s(t)^2 dt = \varepsilon_s.$$

Note that Parseval's theorem states that, the signal energy is, $\int_{-\infty}^{\infty} |s(t)|^2 dt = \int_{-\infty}^{\infty} |S(f)|^2 df$.

Hence, the output power (of the signal sampled at $t = T$) is $P_s = \varepsilon_s^2$. The noise PSD at the output of the matched filter is

$$\Phi_o(f) = \frac{N_0}{2}|H(f)|^2$$

Hence, the average noise power is,

$$P_n = \int_{-\infty}^{\infty} \Phi_o(f)df = \frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2 df = \frac{N_0}{2}\int_{-\infty}^{\infty} |S(f)|^2 df = \frac{1}{2}\varepsilon_s N_0$$

Therefore, the output SNR is,

$$\text{SNR}_o = \frac{P_s}{P_n} = \frac{\varepsilon_s^2}{\frac{1}{2}\varepsilon_s N_0} = \frac{2\varepsilon_s}{N_0}$$

11.2 Detector

- Either correlator or matched filter produces sampled output vector $\mathbf{r} = [r_1 \quad r_2 \quad ... \quad r_N]$ which contains all the relevant information in the received signal waveform.
- Design a detector that makes a decision on $s_m(t)$ in each symbol interval T based on observing $\mathbf{r}$.
- The criterion is to maximize the (probability of) correct decision in the presence of noise.

11.2.1 MAP, ML and MD Detectors

Maximum Aposteriori Probability (MAP) Criterion:

Let us compute the aposteriori probabilities, $P(\mathbf{s}_m|\mathbf{r}), \forall m$, and select the maximum of the set $\{P(\mathbf{s}_m|\mathbf{r})\}$

Using Bayes' rule,

$$P(\mathbf{s}_m|\mathbf{r}) = \frac{P(\mathbf{r}|\mathbf{s}_m)P(\mathbf{s}_m)}{P(\mathbf{r})}$$

where $P(\mathbf{s}_m)$ is apriori probability of m^{th} signal being transmitted. Also,

$$P(\mathbf{r}) = \sum_{m=1}^{M} P(\mathbf{r}|\mathbf{s}_m)P(\mathbf{s}_m).$$

Maximum Likelihood (ML) Criterion:

Let us suppose that the M signals are equiprobable, $P(\mathbf{s}_m) = 1/M$. Note that $P(\mathbf{r})$ is independent of which signal was transmitted. Therefore, the decision rule based on finding the signal that maximizes $P(\mathbf{s}_m|\mathbf{r})$ is <u>equivalent</u> to finding the signal that maximizes $P(\mathbf{r}|\mathbf{s}_m)$

Minimum Distance (MD) Criterion:

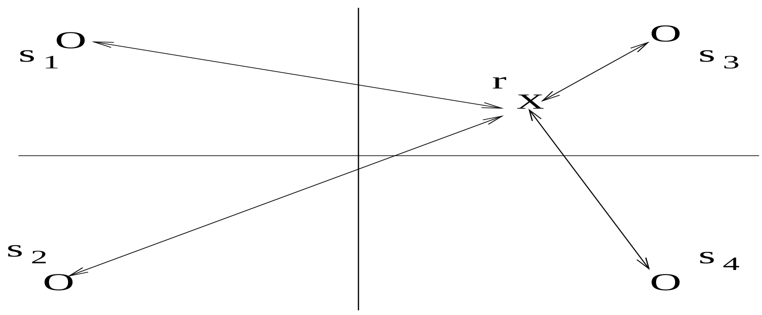
In the presence of AWGN, the joint conditional probability $P(\mathbf{r}|\mathbf{s}_m)$ is given by

$$P(\mathbf{r}|\mathbf{s}_m) = \frac{1}{(\pi N_0)^{N/2}} \exp\left[-\sum_{k=1}^{N} \frac{(r_k - s_{mk})^2}{N_0}\right]$$

$$\Rightarrow \ln \mathrm{P}(\mathbf{r}|\mathbf{s}_m) = -\frac{N}{2} \ln\,(\pi N_0) - \frac{1}{N_0}\sum_{k=1}^{N}(r_k - s_{mk})^2$$

Therefore, the maximum of $P(\mathbf{r}|\mathbf{s}_m)$ over s_m is equivalent to finding the signal that minimizes the Euclidean distance, $D(\mathbf{r}, \mathbf{s}_m) = \sum_{k=1}^{N}(r_k - s_{mk})^2$.

♠ $\pi/4$-shifted QPSK with $r = [0.5 \quad 0.8]$. What is $D(r, s_1)$?



Signals as vectors: Optimum for AWGN channel.

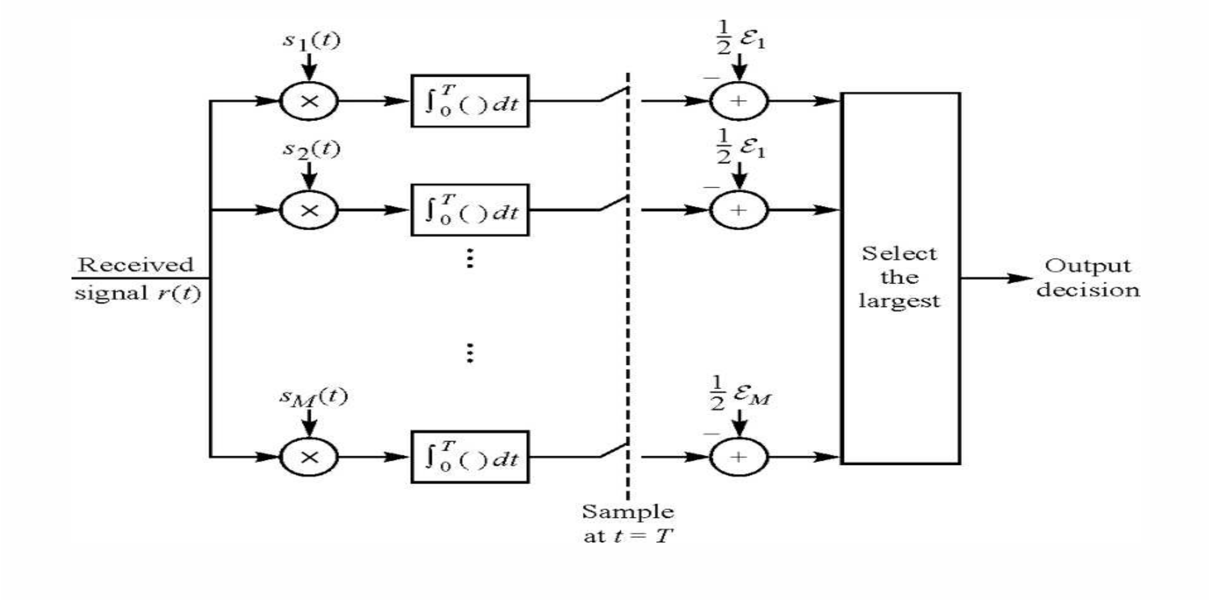
$$D(\mathbf{r}, \mathbf{s}_m) = \sum_{k=1}^{N} r_k^2 - 2\sum_{k=1}^{N} r_k s_{mk} + \sum_{k=1}^{N} s_{mk}^2$$

$$\Rightarrow \| \mathbf{r} \|^2 - 2\mathbf{r}.\mathbf{s}_m + \| \mathbf{s}_m \|^2$$

Therefore, let us minimize $-2\mathbf{r}.\mathbf{s}_m + \| \mathbf{s}_m \|^2$ or maximize $\mathbf{r}.\mathbf{s}_m - \frac{\|\mathbf{s}_m\|^2}{2}$

In the presence of AWGN, the optimum ML detector computes

- a set of M **D**istance metrics, or
- a set of M **C**orrelation metrics



and selects the signal corresponding to the smallest and the largest metric respectively.

Note that the above development of optimum detector (ML and MD) assumes equiprobable symbols. Otherwise, the optimum MAP detector bases its decision on $P(\mathbf{s}_m|\mathbf{r}), m = 1, 2, ..., M$, or equivalently $P(\mathbf{r}|\mathbf{s}_m)P(\mathbf{s}_m)$ since $P(\mathbf{r})$ is irrelevant in the optimization.

11.3 MAP Detector for Non Equiprobable Symbols

Binary PAM signals with signal points, $s_1 = -s_2 = \sqrt{\varepsilon_b}$. Let $P(s_1) = p$ and $P(s_2) = 1 - p$. Signals are corrupted by AWGN with PSD of $\frac{N_0}{2}$.

The received signal vector for binary PAM is, (note that N = 1),

$$r = \pm\sqrt{\varepsilon_b} + n.$$

We have seen that n is a zero-mean Gaussian RV with variance $\sigma_n^2 = \frac{N_0}{2}$.

Therefore, the conditional PDFs are

$$P(r|s_1) = \frac{1}{\sqrt{2\pi}\sigma_n} \exp\left[-\frac{(r-\sqrt{\varepsilon_b})^2}{2\sigma_n^2}\right]$$

$$P(r|s_2) = \frac{1}{\sqrt{2\pi}\sigma_n} \exp\left[-\frac{(r+\sqrt{\varepsilon_b})^2}{2\sigma_n^2}\right]$$

Let $\beta() \equiv P(\mathbf{r}|\mathbf{s}_m)P(\mathbf{s}_m)$. Based on MAP criterion, the decision rule is

$$\frac{\beta(r,s_1)}{\beta(r,s_2)} \gtrless_{s_2}^{s_1} 1.$$

Since

$$\frac{\beta(r,s_1)}{\beta(r,s_2)} = \frac{p}{1-p} \exp\left[\frac{(r+\sqrt{\varepsilon_b})^2-(r-\sqrt{\varepsilon_b})^2}{2\sigma_n^2}\right]$$

$$\Rightarrow r \gtrless_{s_2}^{s_1} \frac{N_0}{4\sqrt{\varepsilon_b}} \ln\left(\frac{1-p}{p}\right)$$

- Since $C(\mathbf{r},\mathbf{s}_1) = r\sqrt{\varepsilon_b}$, the optimum detector compares the correlation metric with the threshold $\chi(p,N_0) = \frac{N_0}{4}\ln\left(\frac{1-p}{p}\right)$ and decides as

$$C(r,s_1) \gtrless_{s_2}^{s_1} \chi(p,N_0)$$

- The decision is influenced by noise PSD, pulse energy and symbol probability when symbols are not equiprobable.

- When $p = \frac{1}{2}$, $\chi(.) = 0$ and in this case, minimum Euclidean distance criterion can be used.

♠ Consider $\frac{\pi}{4}-$shifted QPSK and its constellations. What are the decision regions assuming equiprobable symbols?

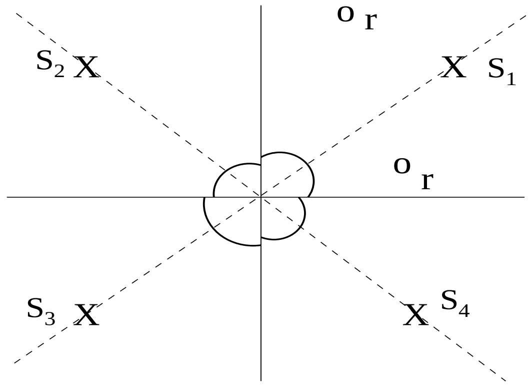
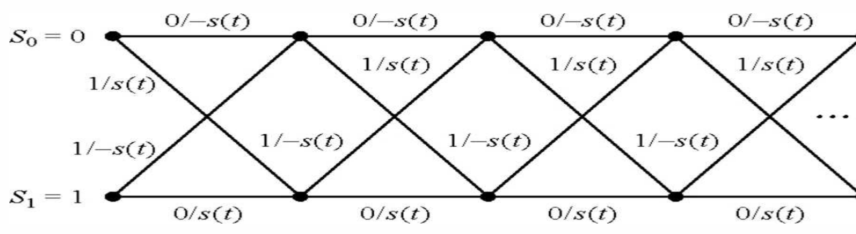


Figure 3: $\frac{\pi}{4}$−shifted QPSK Constellations and Decision Regions for Equiprobable Symbols

11.4 Maximum-Likelihood Sequence Detector (MLSD)

When the signals transmitted in successive symbol intervals are interdependent (i.e., with memory), the optimum detector bases its decisions on the observation of a sequence of received signals over successive intervals. MLSD searches for the minimum Euclidean distance path through the trellis which characterize the memory.



Let us consider an example with NRZI signal whose memory is shown in the trellis for binary PAM signaling scheme.

The output of the matched-filter (or correlation modulator) in the l^{th} symbol interval is

$$r_l = \pm\sqrt{\varepsilon_b} + n_l$$

Let us now consider a sequence of outputs $\{r_1, r_2, ..., r_L\}$. The noise sequence $\{n_1, n_2, ..., n_L\}$ is white and Gaussian. Therefore, for any transmitted sequence $\mathbf{s}^{(m)}$, L marginal PDFs are

$$P(r_1, r_2, ..., r_L|\mathbf{s}^{(m)}) = \prod_{l=1}^{L} P(r_l|s_l^{(m)})$$

$$= \frac{1}{(2\pi\sigma_n^2)^{L/2}} \exp\left[-\sum_{l=1}^{L} \frac{(r_l - s_l^{(m)})^2}{2\sigma_n^2}\right]$$

with $s_l = \pm\sqrt{\varepsilon_b}$.

Therefore, with the received sequence $\{r_1, ..., r_L\}$, the detector determines the sequence $\mathbf{s}^{(m)} = \{s_1^{(m)}, ..., s_L^{(m)}\}$ that maximizes the above conditional PDF (maximum-likelihood sequence detector).

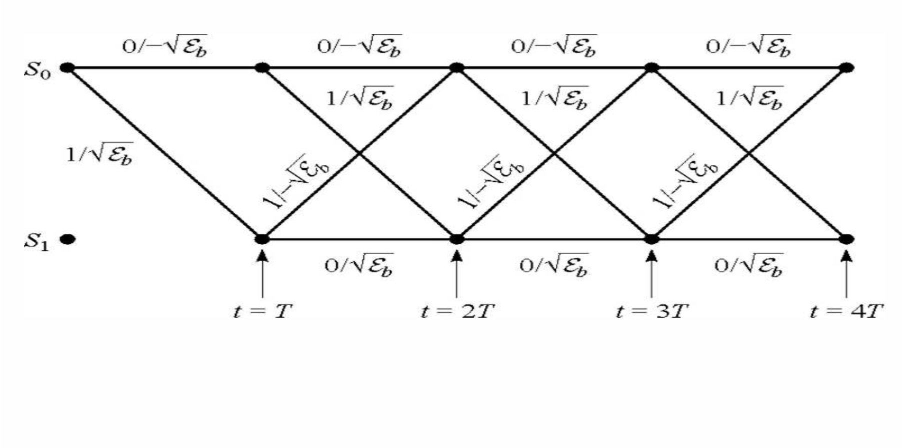
Equivalently, the above MLSD selects the sequence $\mathbf{s}^{(m)}$ that minimizes the Euclidean metric,

$$D(\mathbf{r}, \mathbf{s}^{(m)}) = \sum_{l=1}^{L} (r_l - s_l^{(m)})^2.$$

In general, a total of M^L sequences is possible. However, the number of sequence search in the trellis can be reduced using Viterbi algorithm.

Viterbi algorithm - a trellis search algorithm for performing maximum likelihood sequence detection

At steady state (and thereafter), at $t = nT, n \geq 2$, there are two paths entering and leaving each node.



For example, at $t = 2T$, two paths entering node s_0 corresponding to the information bits $(0, 0)$ and $(1, 1)$. The Euclidean distances are

$$D_0(0, 0) = (r_1 + \sqrt{\varepsilon_b})^2 + (r_2 + \sqrt{\varepsilon_b})^2$$

$$D_0(1, 1) = (r_1 - \sqrt{\varepsilon_b})^2 + (r_2 + \sqrt{\varepsilon_b})^2$$

- Survivor paths and metrics are recorded at each node.
- Add-Compare-Select (ACS) operation is done along the trellis.

12 Performance Evaluation in AWGN Channel

12.1 Error Probability

Assume that optimum receiver is used in the following.

12.1.1 Binary PAM

Let us consider equiprobable symbols with signals $s_1(t)$ and $s_2(t)$. We have seen in AWGN channel,

$$P(r|s_1) = \frac{1}{\sqrt{\pi N_0}} e^{-(r-\sqrt{\varepsilon_g})^2/N_0}$$

Given that $s_1(t)$ was transmitted,

$$P(e|s_1) = \int_{-\infty}^{0} P(r|s_1)dr = \frac{1}{\sqrt{\pi N_0}} \int_{-\infty}^{0} e^{-(r-\sqrt{\varepsilon_g})^2/N_0} dr$$

$$P(e|s_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\sqrt{2\varepsilon_g/N_0}} e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}} \int_{\sqrt{2\varepsilon_g/N_0}}^{\infty} e^{-x^2/2} dx = Q\left(\sqrt{\frac{2\varepsilon_g}{N_0}}\right)$$

where $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{t^2/2} dt$.

Similarly, $P(e|s_2)$ can be computed. Hence, the average probability of error for equiprobable equal energy binary PAM signals (i.e., antipodal signals) is

$$P_b^{\text{antipodal}} = \frac{1}{2}P(e|s_1) + \frac{1}{2}P(e|s_2) = Q\left(\sqrt{\frac{2\varepsilon_g}{N_0}}\right) = Q(\sqrt{2\gamma_b})$$

where γ_b is the SNR per bit.

The probability of error does not depend on the shape of the transmitted signal rather depends on its energy! How about bandwidth of the signal?

We noted that for binary PAM, $d_{12} = 2\sqrt{\varepsilon_g}$. Hence, in terms of distance between signals,

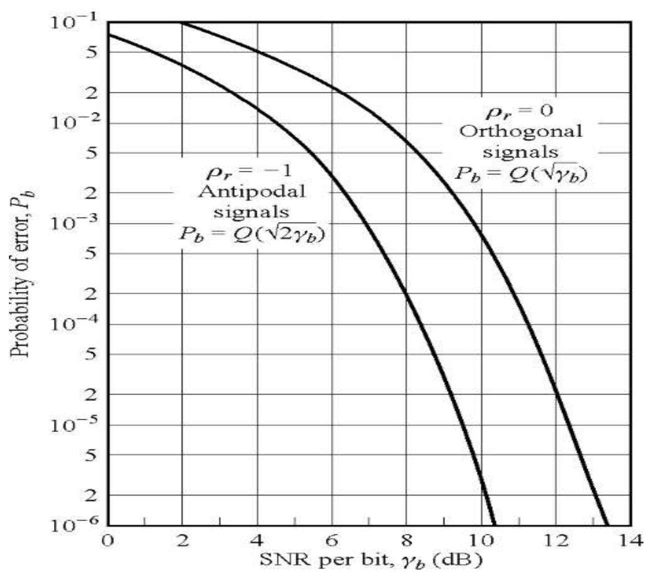
$$P_b = Q\left(\sqrt{\frac{d_{12}^2}{2N_0}}\right)$$

Let us now compare the performance of antipodal signals with orthogonal signals using the distance metrics. For binary orthogonal signals with equal energy, $d_{12} = \sqrt{2\varepsilon_g}$.

Hence,

$$P_b^{\text{orthogonal}} = Q\left(\sqrt{\frac{d_{12}^2}{2N_0}}\right) = Q\left(\sqrt{\frac{\varepsilon_g}{N_0}}\right) = Q(\sqrt{\gamma_b})$$

The larger the argument the smaller the P_b in Q(.) function. Therefore, orthogonal signals are 3 dB poorer than antipodal signals. That is, in order to get the same error performance, the orthogonal signals have to increase their energy by a factor of 2 compared to antipodal signals.



12.1.2 M-ary Orthogonal Signals

For equal-energy (ε_s) orthogonal signals, the optimum detector selects the signal with the largest cross correlation between the received vector **r** and each of the M possible transmitted signal vector $\{\mathbf{s}_m\}$.

$$C(\mathbf{r},\mathbf{s}_m) = \mathbf{r}.\mathbf{s}_m = \sum_{k=1}^{M} r_k s_{mk}, \qquad m = 1, 2, ..., M.$$

Let us suppose that the signal $\mathbf{s}_1$ was transmitted. Therefore, the received signal vector is,

$$\mathbf{r} = [\sqrt{\varepsilon_s} + n_1 \quad n_2 \quad n_3 \quad ... \quad n_M]$$

The output of the bank of M correlators are,

$$C(\mathbf{r}, \mathbf{s}_1) = \sqrt{\varepsilon_s}(\sqrt{\varepsilon_s} + n_1), \qquad \text{and}$$

$$C(\mathbf{r}, \mathbf{s}_m) = \sqrt{\varepsilon_s} n_m, \qquad m = 2, ..., M$$

Therefore, the PDF of the first correlator output (after normalization by $\sqrt{\varepsilon_s}$) is,

$$P_{r_1}(x_1) = \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{(x_1 - \sqrt{\varepsilon_s})^2}{N_0}\right]$$

The PDFs of the other $M-1$ correlator outputs are

$$P_{r_m}(x_m) = \frac{1}{\sqrt{\pi N_0}} e^{-x_m^2/N_0}$$

The average probability of symbol error (P_M) can be computed as follows: Let the probability of correct decision be P_c. Then, $P_M = 1 - P_c$.

$$P_c = \int_{-\infty}^{\infty} p(n_2 < r_1, ..., n_M < r_1) p(r_1) dr_1$$

Since $\{n_m\}$'s are statistically independent Gaussian RVs,

$$P_c = \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{r_1\sqrt{2/N_0}} e^{-x^2/2} dx \right)^{M-1} p(r_1) dr_1$$

where $P(n_m < r_1 | r_1) = \int_{-\infty}^{r_1} P_{r_m}(x_m) dx_m = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{r_1\sqrt{2/N_0}} e^{-x^2/2} dx$.

Symbol and Bit Error Probability

- In comparing different modulation schemes, it is desirable to work with SNR per bit (rather per symbol).

- Since $M = 2^k$, $\varepsilon_s = k\varepsilon_g$.
- It is desirable to convert probability of symbol error P_M, into probability of bit error, P_b.

For equiprobable orthogonal signals, all symbol errors occur with equal probability of,

$$\frac{P_M}{M-1} = \frac{P_M}{2^k - 1}$$

$$\spadesuit P(s_m|s_1) = \frac{P_M}{M-1}. \text{ For } M = 2, P_M = P_b.$$

There are $\binom{k}{n}$ ways in which n bits out of k, may be in error. Therefore, the average number of bit errors per $k-$bit symbol is,

$$P_b = \sum_{n=1}^{k} n \binom{k}{n} \frac{P_M}{2^k - 1} = \frac{k2^{k-1}}{2^k - 1} P_M.$$

Hence, the average bit error probability is $P_b = \frac{2^{k-1}}{2^k-1} P_M \approx \frac{P_M}{2}, \quad k \gg 1.$

12.1.3 M-ary PAM

Consider the carrier-modulated case. Recall: $s_m = \sqrt{\frac{\varepsilon_g}{2}} A_m$ and $A_m = (2m - 1 - M)d$. $d_{\text{min}} = d\sqrt{2\varepsilon_g}$. Assuming equiprobable symbols, the average energy is,

$$\varepsilon_{\text{av}} = \frac{1}{M} \sum_{m=1}^{M} \varepsilon_m = \frac{d^2}{6}(M^2 - 1)\varepsilon_g$$

- The optimum detector compares the demodulator output $r = s_m + n$ with the set of $M - 1$ thresholds.
- With equiprobable symbols, the decision thresholds are the mid-points between two signal points.

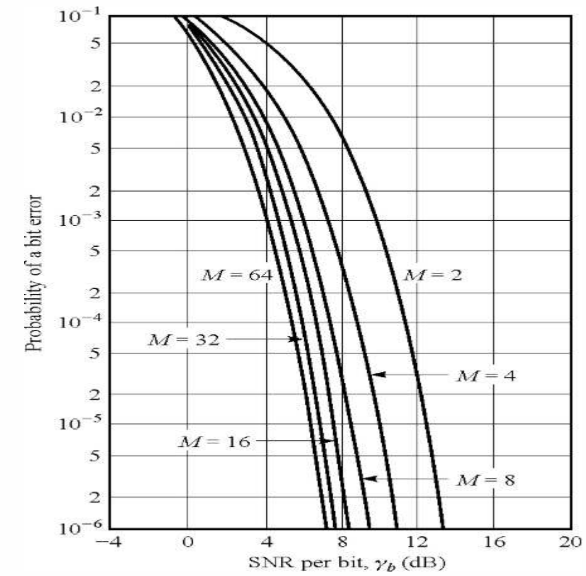


Figure 4: Probability of bit error for orthogonal signals.

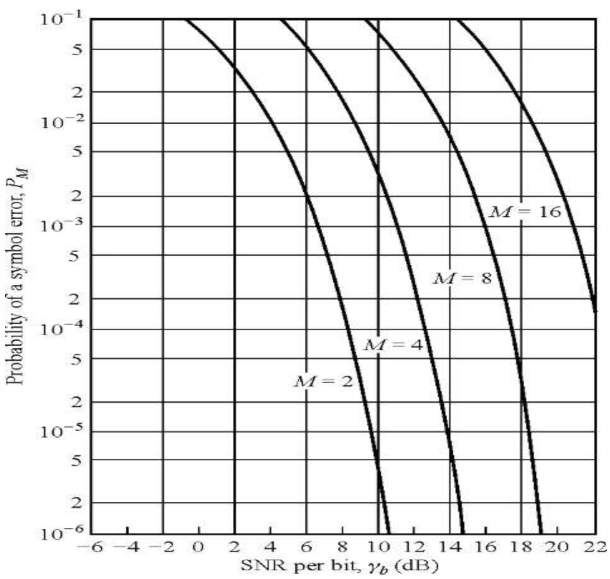


Figure 5: Probability of symbol error for PAM signals.

- The decision is made in favor of the amplitude level closer to r.
- Therefore, average symbol error probability is,

$$P_M = \frac{M-2}{M} P\left(|r - s_m| > d\sqrt{\frac{\varepsilon_g}{2}}\right) + \frac{1}{M} P\left(r > s_m + d\sqrt{\frac{\varepsilon_g}{2}}\right) + \frac{1}{M} P\left(r < s_m - d\sqrt{\frac{\varepsilon_g}{2}}\right)$$

$$P_M = \frac{M-1}{M} P(|r - s_m| > d\sqrt{\varepsilon_g/2})$$

$$P_M = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{d^2 \varepsilon_g}{N_0}}\right)$$

With $k = \ln(M)$, the symbol error can be expressed in terms of SNR per bit ($\varepsilon_{g,\text{av}}/N_0$) as,

$$P_M = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6 \ln(M)}{(M^2-1)} \frac{\varepsilon_{g,\text{av}}}{N_0}}\right),$$

where $\varepsilon_{\text{av}} = \ln(M)\varepsilon_{g,\text{av}}$ and $\varepsilon_g = \frac{6\varepsilon_{\text{av}}}{(M^2-1)d^2}$.

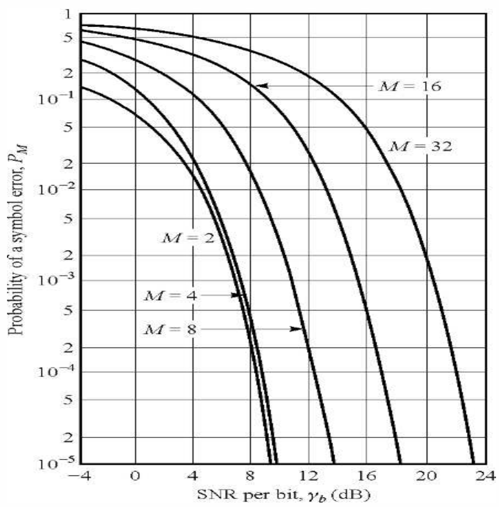


Figure 6: Probability of symbol error for PSK signals.

♠ As M is increased, the effect in symbol error probability for FSK and PSK?

12.1.4 M-ary PSK

- Note that the phase-modulated signal waveforms may be expressed as

$$s_m(t) = g(t)\cos\left[2\pi f_c t + \frac{2\pi}{M}(m-1)\right].$$

- The vector representation is

$$\mathbf{s}_m = \left[\sqrt{\varepsilon_s}\cos\frac{2\pi}{M}(m-1) \quad \sqrt{\varepsilon_s}\sin\frac{2\pi}{M}(m-1)\right]$$

$\varepsilon_s = \varepsilon_g/2$.

- Phase detection problem in PSK.

Since the signal waveforms have equal energy (and assuming equiprobable symbols), the optimum detector for AWGN channel computes the correlation metrics, $C(\mathbf{r}, \mathbf{s}_m)$. The (corrupted) received signal vector is $\mathbf{r} = [r_1 \quad r_2]$. The phase of $\mathbf{r}$ is, $\theta_r = \tan^{-1}(\frac{r_2}{r_1})$. We need to compute the PDF of θ_r to compute P_M.

$$P_M = 1 - \int_{-\pi/M}^{+\pi/M} p_{\theta_r}(\Theta_r)d\Theta_r$$

In general, the integral can only be evaluated numerically except for $M = 2, 4$.

♠ For M=2 (antipodal signals) $P_2 = Q(\sqrt{2\gamma_b})$. For $M = 4$,

$$P_4 = 1 - (1-P_2)^2 = 2Q(\sqrt{2\gamma_b})\left[1 - \frac{1}{2}Q(\sqrt{2\gamma_b})\right]$$

12.2 Bandwidth Requirement

One can compare the SNR required to achieve a specified P_e for different modulation schemes. However, it is not just enough or fair. Bandwidth (and achievable data rate) requirement has also to be taken into account.

12.2.1 Bandwidth Limited Case

For multi-phase signals (such as PSK, PAM), the bandwidth requirement is the bandwidth of the equivalent lowpass signal, $g(t)$.

Let us suppose that the bandwidth of signal $g(t)$ is $W = \frac{1}{T}$ (approximately). Since $T = \frac{k}{R}$ and $k = \ln(M)$,

$$W = \frac{R}{\ln M}.$$

The bandwidth efficiency is measured in terms of bit/s/Hz. Hence, as M is increased, when R is fixed, the bandwidth requirement decreases.

Modulated PAM: SSB and DSB

The bandwidth efficient method to transmit PAM signal is to use SSB modulation with bandwidth of $\frac{1}{2T}$. Hence, the efficiency is $2 \times \ln(M)$. This is 3 dB better than DSB.

12.2.2 Power Limited Case

Orthogonal Signals

Let us construct M orthogonal FSK signals with minimum frequency separation of $\frac{1}{2T}$. Hence, the bandwidth requirement is,

$$W = \frac{M}{2T} = \frac{M}{2(k/R)} = \frac{M}{2\ln(M)}R$$

In this case, as M is increased, when R is fixed, the bandwidth requirement increases.

The meaningful comparison is based on the normalized data rate (R/W) versus the SNR per bit (ε_g/N_0) required to achieve a given error probability as shown in the figure.

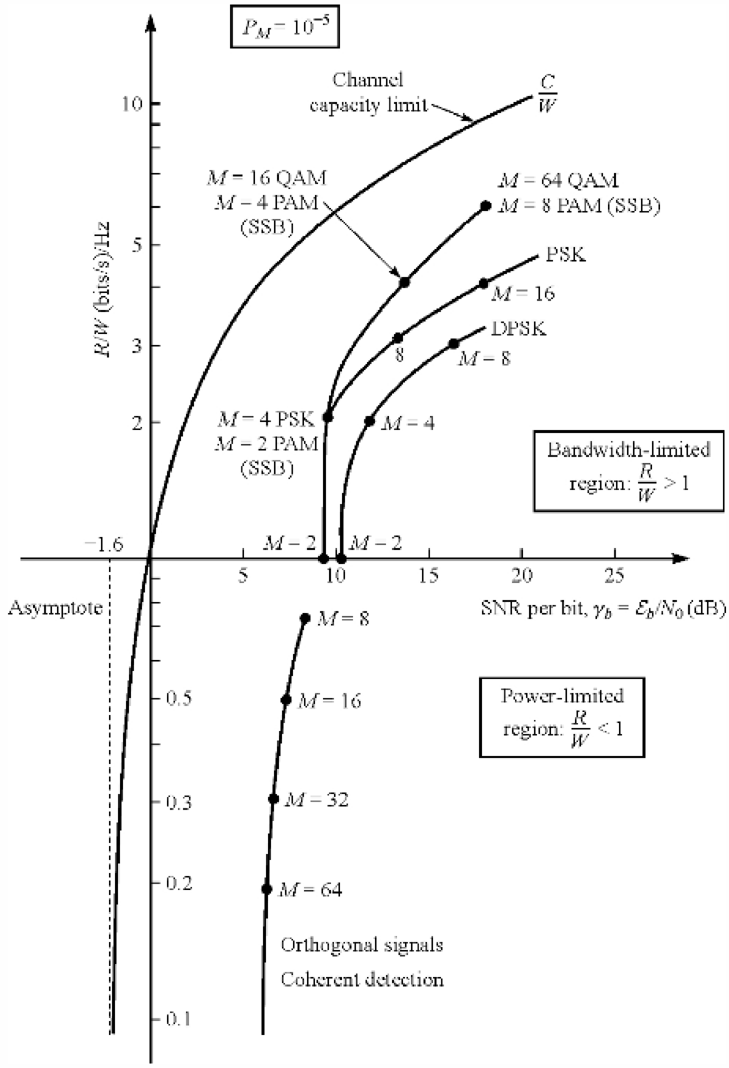


Figure 7: Comparison of several modulation methods at 10^{-5} symbol error probability.

- In the case of PAM, QAM and PSK, increasing M results in higher ratio of R/W with the cost of increased SNR per bit (i.e., signal power).
- In M-ary orthogonal signals, increasing M results in lower ratio of R/W with the benefit of decreased SNR per bit.

- The maximum possible achievable ratio of C/W (where $C = R$ bit/s) is Shannon capacity limit for AWGN channel. It is an upperbound on the bandwidth efficiency of any type of modulation (either band-limited or power-limited case).

12.3 Analog vs Digital Repeaters

We noted that in digital transmission through AWGN channel, performance in terms of probability of error (either symbol or bit) depends solely on the received SNR, $\frac{\varepsilon_g}{N_0}$ where the transmitted signal energy is ε_g.

In addition to additive noise, channel attenuation also affects the performance. Hence, the received signal can be,

$$r(t) = \alpha s(t) + n(t), \qquad 0 < \alpha \leq 1.$$

Therefore, the received signal energy is $\alpha^2 \varepsilon_g$. Consequently, the received SNR is $\left(\frac{2\varepsilon_g}{\frac{N_0}{\alpha^2}}\right)$.

Analog Repeaters

- Each repeater (amplifier) boosts the signal by multiplying $\frac{1}{\alpha}$, where $\frac{1}{\alpha} \geq 1$.
- Amplifiers boost both signal and noise.
- If K analog repeaters are used, the bit-error probability for binary PAM transmitted signal is given as

$$P_b^{\text{analog}} \approx Q\left(\sqrt{\frac{2\varepsilon_g}{KN_0}}\right),$$

  where $\alpha >> 0$ and the output SNR at the last repeater can be approximated as $\left(\frac{2\varepsilon_g}{KN_0}\right)$.

Regenerative Repeaters

- No boosting of signals in digital transmission.
- Instead, demodulator/detector demodulates/detects the transmitted symbol sent by the preceding repeater and then corresponding waveform is transmitted to the next repeater.
- Therefore, additive noise does not accumulate.
- The only error that occurs is in the detection of the symbol and this error may propagate up to the last repeater.
- Suppose that the binary PAM is used. Hence, the probability of (bit) error is,

$$P_b = Q\left(\sqrt{\frac{2\varepsilon_g}{N_0}}\right)$$

  Note that the output SNR of any repeater is $(\frac{2\varepsilon_g}{N_0})$.

- Assuming a symbol is incorrectly detected only once during the transmission through a bank of K repeaters,

$$P_b^{\text{digital}} \approx KQ\left(\sqrt{\frac{2\varepsilon_g}{N_0}}\right).$$

Digital regenerative repeaters offer significant power savings (or equivalently the number of repeaters is reduced). Note that, for a given probability of error with $K >> 1$,

$$P_b^{\text{analog}} \approx Q\left(\sqrt{\frac{2\varepsilon_g}{KN_0}}\right) >> P_b^{\text{digital}} \approx KQ\left(\sqrt{\frac{2\varepsilon_g}{N_0}}\right).$$

13 Signal Design for Band-limited Channels

- Bandwidth constraint of the channel has to be taken into account in the design of pulses for transmission of digital information sequences.
- Let us model a channel as a band-limited linear filter having an equivalent lowpass frequency response of $C(f)(=0, |f| > W)$.
- How to design a pulse $g(t)$ to transmit through the (AWGN) channel with minimum distortion?

In general,

$$C(f) = |C(f)|e^{j\theta(f)}$$

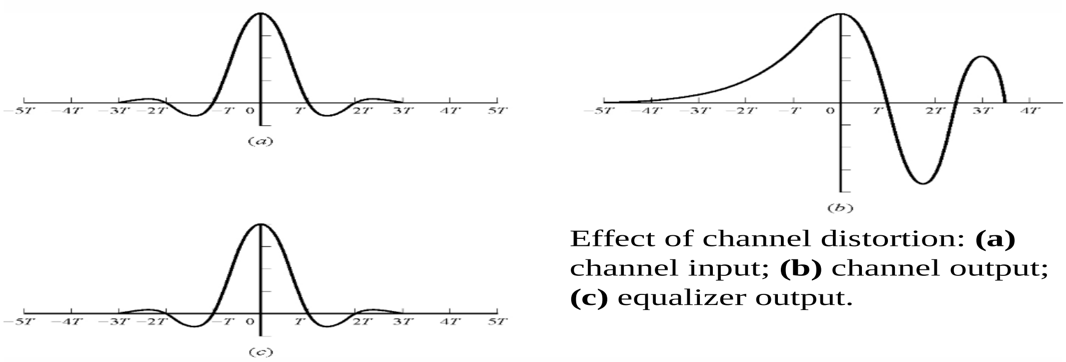
where $|C(f)|$ and $\theta(f)$ are amplitude and phase responses respectively.

- If $|C(f)|$ is constant and $\theta(f)$ is linear (within the bandwidth of interest), it is an ideal (filter) channel.
- Non-ideal channel characteristic causes distortion in amplitude and delay, resulting in interference of signals from different symbol intervals.

♠ As an example, consider a signal transmitted using the (band-limited) pulse $g(t)$ as shown in figure (a).

- One can transmit a sequence of pulses every T seconds since $g(t)$ has zeros at $\pm T, \pm 2T, ...$
- However, non-ideal channel will cause inter symbol interference (ISI) as seen in figure (b).

- Compensation for non-ideal frequency response characteristics can be done using equalizers.



Effect of channel distortion: **(a)** channel input; **(b)** channel output; **(c)** equalizer output.

13.1 Inter Symbol Interference (ISI)

A (linearly) modulated signal (such as PAM or PSK) can be represented as

$$v(t) = \sum_n I_n g(t - nT) \tag{1}$$

where I_n is real and complex for PAM and PSK respectively.

It will be seen that when the channel is ideal, $g(t)$ can be designed to efficiently transmit the symbols (i.e., without ISI). However, with non-ideal channel, ISI among adjacent symbols occurs at higher rates. Also note that in addition to ISI, channel noise $n(t)$ will also cause errors in the received signal.

- Let us suppose that $g(t)$ has a band-limited frequency response of $G(f)(= 0, |f| > W)$. The signal to be transmitted is $v(t)$ as given in (1).

- $v(t)$ is transmitted over the band-limited channel whose lowpass equivalent frequency response is $C(f)$.
- Hence, the received signal is,

$$r(t) = \sum_n I_n h(t - nT) + n(t), \tag{2}$$

where $n(t)$ represents AWGN and,

$$h(t) = \int_{-\infty}^{\infty} g(\tau)c(t-\tau)d\tau.$$

- Suppose that $r(t)$ is passed through a filter and sampled at a rate $1/T$ samples/s.

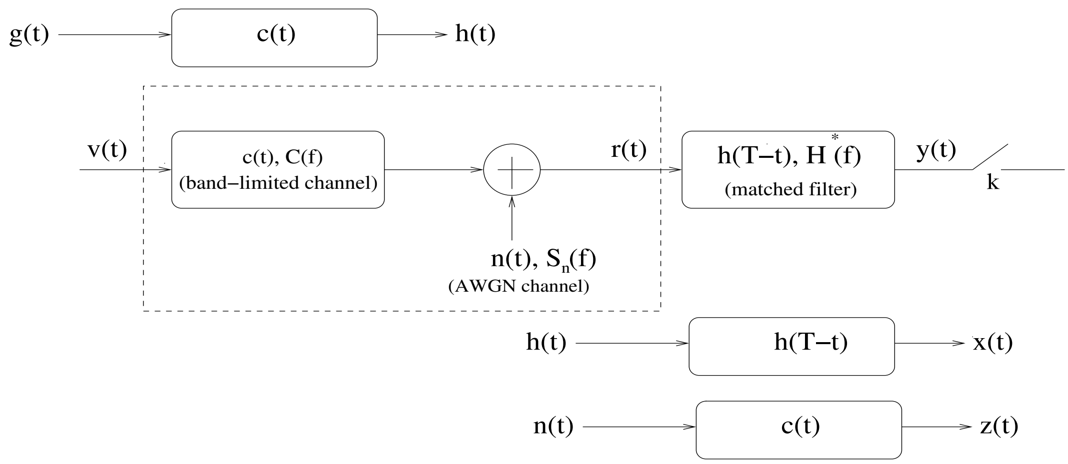


Figure 8: A block diagram.

- In the presence of AWGN, the optimum filter is one that is matched to the signal at the receiver. What is the signal input to the filter in a symbol interval?

Therefore, the output of the receive (matched) filter is,

$$y(t) = \sum_n I_n x(t - nT) + z(t),$$

where $x(t)$ and $z(t)$ are responses of the matched filter to $h(t)$ and $n(t)$ respectively.

Hence, assuming zero transmission delay through the channel, the k^{th} sampled output is,

$$y(kT) \equiv y_k = \sum_n I_n x(kT - nT) + z(kT) = \sum_n I_n x_{k-n} + z_k$$

$$y_k = I_k x_0 + \sum_{n \neq k} I_k x_{k-n} + z_k. \tag{3}$$

The term I_k represents the desired symbol at k^{th} sampling instant and $\sum_{n \neq k} I_n x_{k-n}$ represents ISI. z_k is due to noise.

Eye Diagram

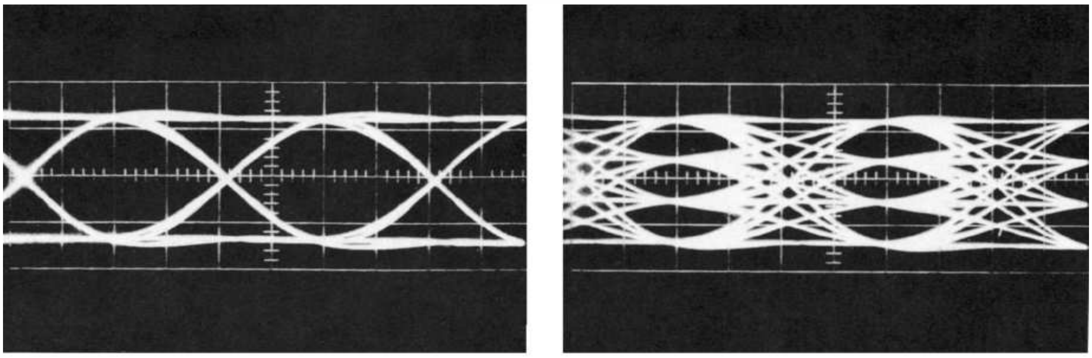
An ISI (+ noise) visualization tool.

The amount of ISI (plus noise) can be viewed on an oscilloscope. The effect of ISI causes the eye to close thereby reducing the margins for the noise to cause errors.

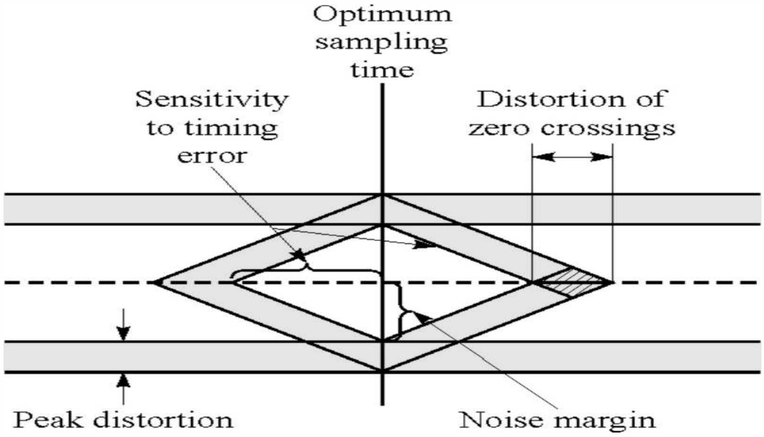
13.2 Nyquist Pulse Design Criterion for No ISI

What should $x(t)$ (or equivalently $g(t)$), be such that there is no ISI. That is, we require (in time domain),

$$\boxed{x(t = kT) = x_k = \begin{cases} 1, & k = 0, \\ 0, & k \neq 0. \end{cases}}$$



(*a*) Binary (*b*) Quaternary



Let us assume an ideal channel, i.e., $C(f) = 1$ for $|f| < W$. Then, by looking at the previous block diagram in Figure 8,

$$X(f) = |H(f)|^2 = |G(f)C(f)|^2 = |G(f)|^2 \tag{4}$$

Hence, from $x(t)$ we can find $X(f)$ and then $G(f)$ and finally $g(t)$!

The necessary and sufficient condition on $X(f)$ in order for $x(t)$ to satisfy the above relationship is also known as Nyquist pulse shaping criterion, which is stated as (in frequency domain),

$$\boxed{B(f) = \sum_{n=-\infty}^{\infty} X(f + n/T) = T}$$

Note that $B(f)$ is a periodic function in f with period $\frac{1}{T}$.

See pp 557 in the text book for proof. Note that

$$b_n = T \int_{-1/2T}^{1/2T} B(f) e^{j2\pi f n T} df \text{ with } B(f) = \sum_{n=-\infty}^{\infty} b_n e^{-j2\pi f n T} \text{ and } \boxed{b_n = T x(-nT)}$$

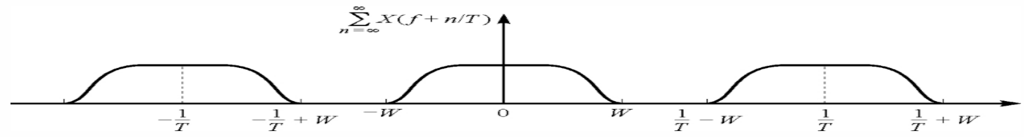
It can be seen from (4), $X(f) = G(f) = 0, |f| > W$, since the channel bandwidth is W, i.e., $C(f) = 0, \quad |f| > W$. The pulse is,

$$g(t) = \int_{-\infty}^{\infty} \sqrt{|X(f)|} e^{j2\pi f t} df$$

A note of caution: There will still be distortion due to noise $n(t)$; in order to minimize the effect due to (AWGN) noise, we used the matched filter.

Case 1: $T < 1/2W$ **or symbol rate,** $\frac{1}{T} > 2W$.

The $\sum_n X(f+n/T)$ consists of non-overlapping $X(f)$ separated by $1/T$. Therefore, there is no choice for $X(f)$ to ensure $B(f) = T$ in this case. Hence, we



cannot design a system with no ISI.

Case 2: $T = 1/2W$ **or symbol rate,** $\frac{1}{T} = 2W$.

When $T = 1/2W$, (i.e., at the Nyquist rate that is required for no frequency-aliasing), the replicas of $X(f)$ are separated by only $1/T$. Hence, there exists only one $X(f)$ that results in $B(f) = T$. That is,

$$X(f) = \begin{cases} T, & |f| < W, \\ 0, & \text{otherwise.} \end{cases}$$

The corresponding pulse is

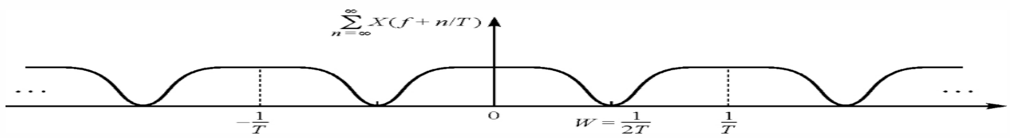
$$x(t) = \text{sinc}\left(\frac{\pi t}{T}\right) = \frac{\sin\,(\pi t/T)}{(\pi t/T)}$$

♠ How about g(t)? $g(t) = \int \sqrt{X(f)} e^{j2\pi ft} df = \int \sqrt{T} e^{j2\pi ft} df = \frac{1}{\sqrt{T}} \text{sinc}(\pi t/T)$.

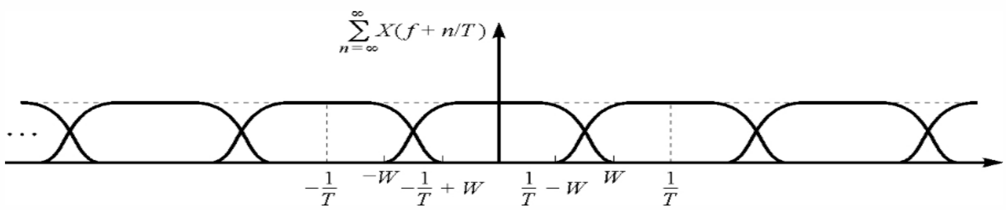
This pulse is • non-causal (and hence non-realizable unless shifted version of it is used), • converges slower and • timing error in sampling also causes problems.

Case 3: $T > 1/2W$ **or symbol rate,** $\frac{1}{T} < 2W$.

In this case, $B(f)$ consists of overlapping replicas of $X(f)$ separated by $1/T$.



There are many pulses such that $B(f) = T$. In particular, the raised cosine pulses (in f domain) are used in practice that have desirable spectral characteristics.



$$X_{rc}(f) = \begin{cases} T, & \left(0 \le |f| \le \frac{1-\beta}{2T}\right) \\ \frac{T}{2}\left\{1 + \cos\left[\frac{\pi T}{\beta}\left(|f| - \frac{1-\beta}{2T}\right)\right]\right\}, & \left(\frac{1-\beta}{2T} \le |f| \le \frac{1+\beta}{2T}\right) \\ 0 & \left(|f| > \frac{1+\beta}{2T}\right) \end{cases}$$

where β is called *roll-off factor* with $0 \leq \beta \leq 1$. The excess bandwidth (in excess of $1/2T$) is expressed as a percentage of the Nyquist bandwidth (i.e., $2W$).

♠ For example, $\beta = 1$ corresponds the bandwidth is $1/T$ and the symbol rate is $\frac{1}{T} = W$ with excess bandwidth of 100%. Time domain signal is,

$$x(t) = \frac{\sin\,(\pi t/T)}{(\pi t/T)} \frac{\cos\,(\pi \beta t/T)}{1 - 4\beta^2 t^2/T^2}$$

Note that $\beta = 0$ corresponds to $x(t) = \text{sinc}(\pi t/T)$ and the bandwidth of 1/2T. The symbol rate is $\frac{1}{T} = 2W$ with no excess bandwidth.

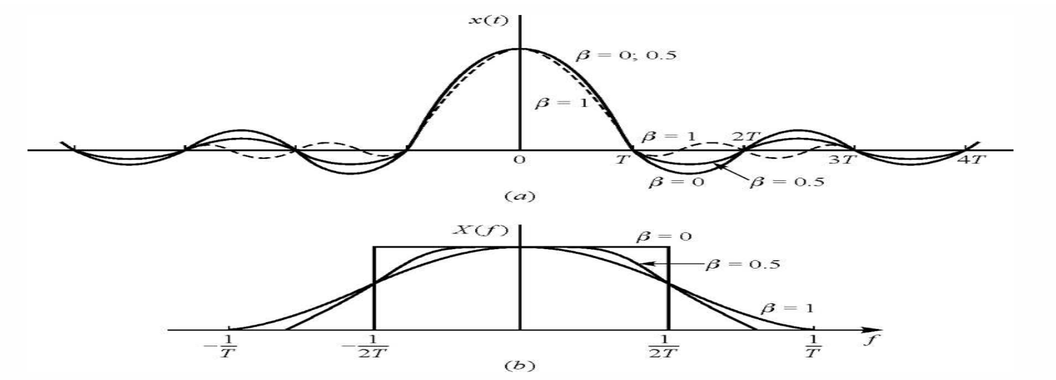


Figure 9: Pulses having a raised cosine spectrum.

Note that $x(t)$ satisfies the time domain constraint (i.e., $x(kT) = 0, k \neq 0$,) for no ISI.

13.3 Band-limited Signal Design with Controlled ISI

- To realize practical filters with zero ISI, $\frac{1}{T} < 2W$. That is, the symbol rate is below the Nyquist rate of 2W samples/s.

- Relax the zero ISI constraint and with controlled ISI, then the symbol rate comparable to Nyquist rate is achievable.

♠ Example with Duobinary Signal Pulse:

$$x(nT) = \begin{cases} 1, & n = 0, 1, \\ 0, & \text{otherwise.} \end{cases}$$

Hence,

$$b_n = \begin{cases} T & n = 0, -1, \\ 0 & \text{otherwise.} \end{cases}$$

Then,

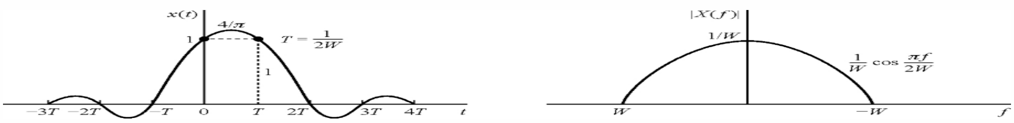
$$B(f) = T + Te^{-j2\pi fT}$$

For $T = 1/2W$,

$$x(t) = \text{sinc}(2\pi Wt) + \text{sinc}[(2\pi(Wt - 1/2)]$$

Note that the spectrum of $x(t)$ decays smoothly and as a result, is realizable. Thus, a symbol rate of 2W samples/s can be achieved.

♠ Example with spectrum shaping: Modified Duobinary Signal Pulse:

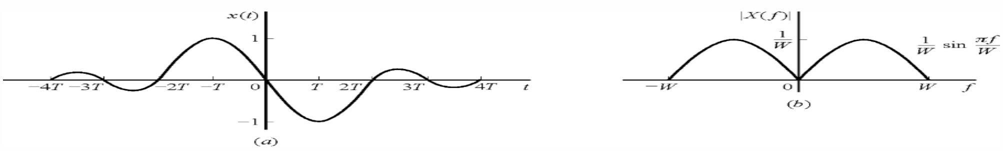


$$x(nT) = \begin{cases} 1, & n = -1, \\ -1, & n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

The corresponding pulse $x(t)$ is

$$x(t) = \text{sinc}\left[\frac{\pi(t+T)}{T}\right] - \text{sinc}\left[\frac{\pi(t-T)}{T}\right]$$

In general, a class of band-limited signal pulses that have the form



$$x(t) = \sum_n x\left(\frac{n}{2W}\right) \text{sinc}\left[2\pi W\left(t - \frac{n}{2W}\right)\right]$$

are called partial-response signals (PRS).

In the above examples, controlled amount of ISI is introduced and symbol transmission of 2W samples/s is achieved within the limited bandwidth of W.

Assignments

Problem Set: 1

1. From the text book (Proakis)
   - **Probability Theory:** 2.1, 2.6, 2.9
   - **Random Process:** 2.11, 2.12, 2.14, 2.16
   - **Signals and Systems:** 4.3, 4.4
   - **Signal Space:** 4.10, 4.12, 4.16

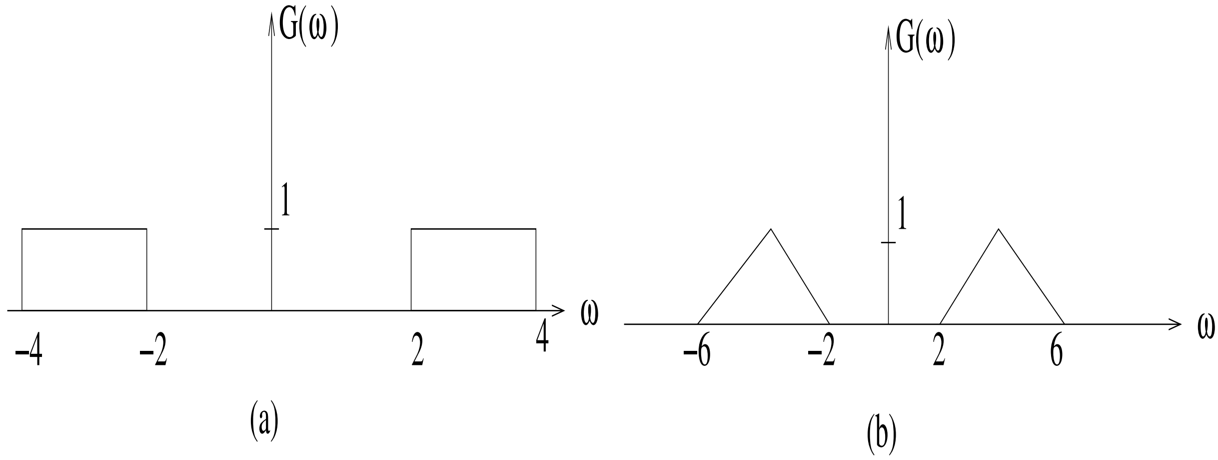
2. Additional Problems:

   Frequency Spectrum: Evaluate the spectra of the signal,

   $$x(t) = \prod(\frac{t-5}{10}) + 8 \sin{(6\pi t)},$$

   where $\prod(\frac{t-t_o}{T})$ is defined as a unit rectangular pulse centered at $t = t_o$ with the width of T.

   Time Domain Signal: Find the inverse Fourier transform of the spectra shown in the figure (use properties of the Fourier transform such as frequency-shifting).



BSC: Over a binary communication channel, the symbol **0** is transmitted with probability 0.4 and **1** is transmitted with probability 0.6. It is known that $P(\epsilon, \mathbf{0}) = 10^{-6}$ and $P(\epsilon, \mathbf{1}) = 10^{-4}$, where $P(\epsilon, x_i)$ is the probability of detecting the error given that x_i is transmitted. Determine $P(\epsilon)$, the error probability of the channel.

Problem Set: 2

1. **Encoding of Information:** From the text book (Proakis), 3.1, 3.2, 3.7; **and**

   A source has an alphabet $\{a_1, a_2, a_3, a_4\}$ with corresponding probabilities $\{0.1, 0.2, 0.3, 0.4\}$.

   (a) Find the entropy of the source?

   (b) What is the minimum required average codeword length to represent this source for error-free reconstruction?

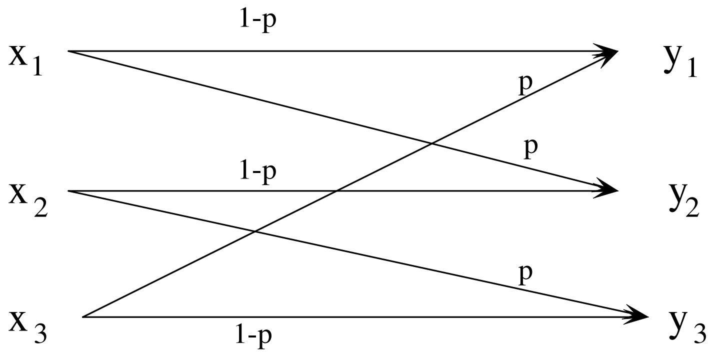
   (c) Design a Huffman code for the source and compare the average length of the Huffman code with the entropy of the source?

   (d) Design a Huffman code for the second extension of the source (take two symbols at a time)? What is the average codeword length? What is the average required binary letters per each source output symbol?

   (e) Compare the efficiency of above coding schemes?

2. **Communication Channel Capacity:** From the text book (Proakis), 7.4, 7.5; **and**

   Determine the capacity of the ternary-input ternary-output channel shown in the figure?



Problem Set: 3

From the text book (Proakis)

- 4.21 (a), (b); 4.22
- 4.29 (a) with $h = \frac{2}{3}$
- Additional problems:

  1. The signal constellation for 16-QAM is shown in Fig. 1. If the transmitted signal is $x(t) = s_4(t) + s_2(t-T) + s_{11}(t-2T)$, where T denotes the symbol duration. Sketch $x(t)$ assuming $f_c = \frac{2}{T}$, where f_c denotes the carrier frequency.

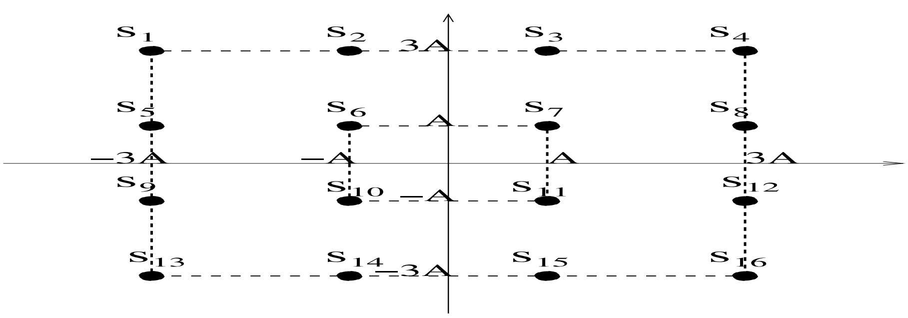


Figure 10: Signal Constellation for 16-QAM.

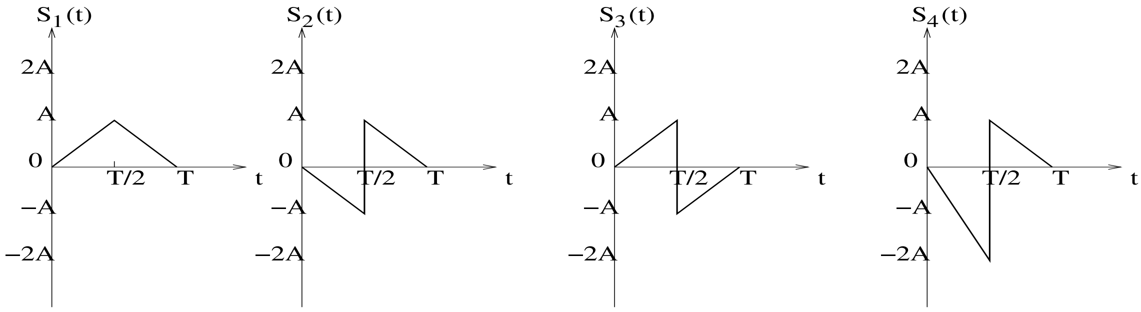


Figure 11: Baseband 4-ary Signalling.

2. A set of transmitted signals in a baseband 4-ary scheme is shown in Fig. 1. (i) Find the average energy $\hat{E}$ of the transmitted signals assuming equiprobable symbols, (ii) Draw the signal constellation for the above signalling scheme.

Problem Set: 4

From the text book (Proakis)

- 5.2, 5.5, 5.8, 5.18 (a), 5.26 (a)

Note that you have to do only the above problems for credit. However, you are encouraged to do the following problems for which the solutions will also be provided.

- 5.19, 5.46, 9.12, 9.16, 9.19

Sample Questions

1. **Probability and Random Process:** (15 pts)

   Determine the mean and the autocorrelation function of the random process

   $$X(t) = A \cos(\omega_c t + \theta),$$

   where θ is an random variable uniformly distributed in the range $(0, 2\pi)$ and A is a constant. Is the process wide-sense stationary ?

2. **Signal and Vector Space:** (15 pts)

   Consider the three waveforms $f_n(t)$ shown in Fig. 1. Are these waveforms are orthogonal ?

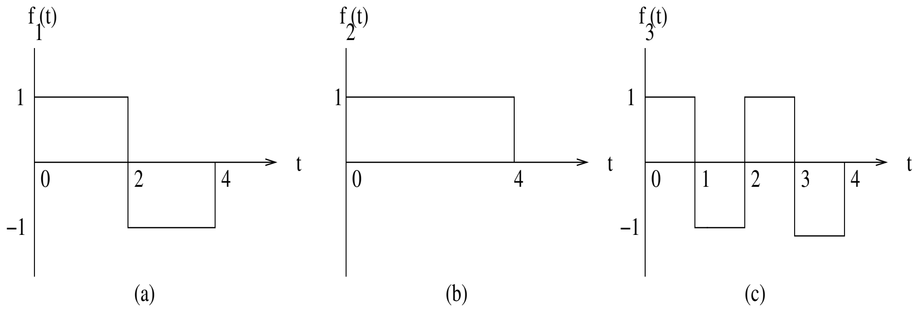


Figure 12: Waveforms, $f_n(t), n = 1, 2, 3.$

3. **Autocorrelation and Power Spectral Density:** (25 pts)

   The white noise process $X(t)$ is the input to the channel whose transfer function is given by,

   $$H(f) = \frac{1}{1 + j2\pi\varrho f},$$

   where ϱ is a constant. The input has the following characteristics: $E[X(t)] = 0$ and autocorrelation function $\phi_{xx}(\tau) = \sigma^2\delta(\tau)$, where $\delta(.)$ is a delta function. The corresponding output process is $Y(t)$. Determine the following:

(a) the output power spectral density $\Phi_{yy}(f)$?

(b) the autocorrelation function of the output process $\phi_{yy}(\tau)$?

(c) the average output power ?

4. **Source Entropy and Block Coding:** (25 pts)

   A discrete memoryless channel has an alphabet of four letters, $x_i, i = 1, 2, 3, 4$, each occurring with probability $\frac{1}{3}, \frac{1}{3}, \frac{1}{6}$ and $\frac{1}{6}$ respectively. Evaluate the efficiency of a fixed-length binary code in which,

   (a) each letter is encoded separately into binary sequence ?

   (b) two letters are encoded separately into binary sequence ?

   (c) from the above results in (a) and (b), what can you conclude about the coding of symbols ?

5. **Noisy Channel Capacity:** (20 pts)

   A communication channel is characterized as a band-limited (to 3 KHz) AWGN waveform channel with noise power spectral density of $N_0 = 8 \times 10^{-8}$ Watts/Hz. The average input power to the channel is 24 mWatts.

   (a) determine the capacity of the channel ?

   (b) is it possible to transmit reliably a signal that has a maximum frequency component at 5 KHz and sampled ar Nyquist rate and encoded with 8 bits/sample? Justify your answer.

6. **Digital Transmission:**

   (a) [**10 pts**] The bandwidth of a baseband channel allows a maximum symbol rate of $R_s = 6 \times 10^3$ symbols/second. Since the source produces a bit rate of $R_b = 20 \times 10^3$ bits/second, M-ary modulation is used. Find M.

(b) **[15 pts]** A video signal is to be stored in a hard drive. The signal is composed of 24 frames/second with 728×512 pixels/frame. The grayness level of each pixel is represented by R bits. The quantization process results in signal-to-noise ratio (SNR) of $\gamma(R) = 1.8 + 6R$ (in dB). If the output quality requirement dictates an SNR of at least 40 dB, determine how many minutes of video signal can be stored in a 40 GB hard drive?

7. **Random Process [20 pts]:** Consider a random process

$$X(t) = A\cos(2\pi f t + \theta_o),$$

where f and θ_o are constants and **A is a uniformly distributed random variable** over the interval (α, β) with $\alpha \neq 0, \beta \neq 0$.

(a) Find E$[X(t)]$.

(b) Find the autocorrelation of $X(t)$, $\phi_X(\tau)$.

(c) Is it possible that $X(t)$ be wide-sense stationary? If so, under what conditions and if not, why?

8. **Signal Space [10 pts]:** Signals $s_1(t)$ and $s_2(t)$ are as shown in Figure 13. Draw two unit-energy signals $s_3(t)$ and $s_4(t)$ which are orthogonal to both $s_1(t)$ and $s_2(t)$.

9. **Filtering of Random Process [25 pts]:** $X(t)$ is a **stationary** random process with power spectral density $S_X(f)$. This process is passed through

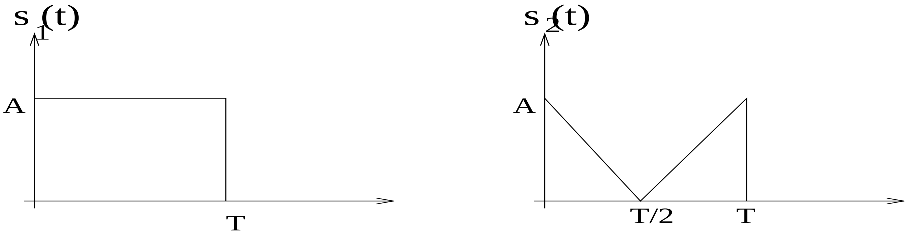


Figure 13: Two signals.

a linear time-invariant system as shown in Figure 14.

(a) Is $Y(t)$ stationary? Justify your answer.

(b) What is the power spectral density of $Y(t)$? Note that, $\frac{d}{dt} \leftrightarrow j2\pi f$.

(c) Given that

$$S_X(f) = \begin{cases} K(1 - f/B), & |f| < B, \\ 0, & \text{otherwise,} \end{cases}$$

where B and K are constants. (i) Plot the output power spectral density and (ii) find the average output power for $B = 10$ kHz and $K = \frac{6\times10^{-12}}{8\pi^2}$.

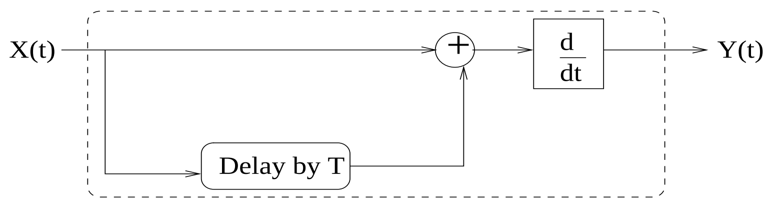


Figure 14: A continuous time LTI system.

10. **Source Coding : [20 pts]** A discrete memoryless source emits messages m_1, m_2 and m_3 with probabilities 0.2, 0.3 and 0.5 respectively.

(a) If two output messages are encoded at a time, determine the optimum code assignment with **fixed** code length?

(b) If two output messages are encoded at a time, determine the optimum code assignment with **variable** code length?

(c) For both cases above, determine the coding efficiency? What can you conclude about the coding of symbols from the efficiency?

11. **Signals and Vectors [15 pts]:** Consider four waveforms as shown in Fig. 15. (i) Show that $\{g_i(t)\}, \quad i = 1, 2, 3,$ forms an orthogonal set. (ii) Find the minimal basis functions for the above set. (iii) Express $x(t)$ shown in Fig. 15 (d) as a vector using the above basis functions.

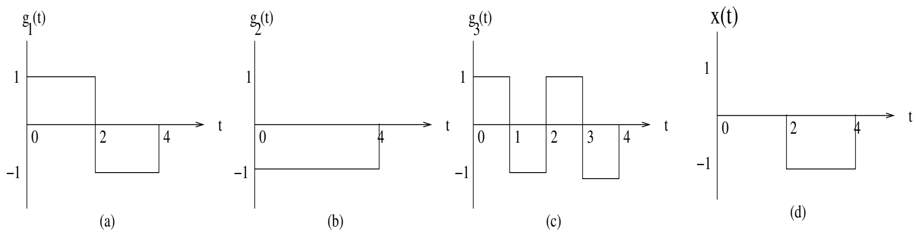


Figure 15: Waveforms.

12. **Channel Capacity [15 pts]:** A communication channel is characterized as a band-limited (to 8 kHz) AWGN waveform channel with noise power spectral density of $N_0 = 10^{-7}$ Watts/Hz. The average input power to the channel is 0.024 Watts.

(a) Determine the capacity of the channel.

(b) Is it possible to transmit**reliably** a baseband signal with bandwidth of 3.5 kHz, sampled at Nyquist rate and encoded with 4 bits/sample?

13. **Random Process [15 pts]:** A random process $v(t)$ is defined as

$$v(t) = X\cos\,(2\pi f_c t) - Y\sin\,(2\pi f_c t),$$

where X and Y are **zero-mean orthogonal** random variables and f_c is a constant. Determine the condition on the statistics of X and Y under which the process $v(t)$ is wide sense stationary. Aid: $\cos\,(A+B) = \cos\,A\,\cos\,B - \sin\,A\,\sin\,B$.

14. (a) **Source Coding: [20 pts]** A binary erase channel (BEC) has the transition probabilities as shown in Fig. 16. Assuming equiprobable inputs, determine (i) the source entropy, $H(X)$ and (ii) conditional mutual information, $H(X|Y)$, (iii) average mutual information $I(X;Y)$, in bits/symbol.

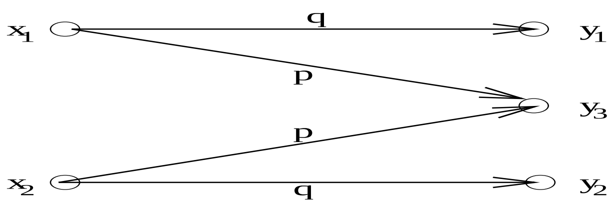


Figure 16: Transition probabilities in BEC.

(b) **[15 pts]:** A discrete memoryless source emits messages m_1 and m_2 with probabilities 0.8 and 0.2 respectively. (i) Find the optimum (Huffman) binary code for this source. (ii) If two output messages are encoded at a time, determine the optimum code assignment. (iii) For both cases above, determine the coding efficiency.

15. **Power Spectral Density [20 pts]:** A noise signal $n_i(t)$ with PSD $S_{n_i}(f) = K$(a constant), for all f, is first filtered using an ideal low pass filter $h(t)$

and then is passed through an ideal differentiator as shown in Fig. 17. The filter response is characterized as: $H(f) = \begin{cases} 1, & |f| \leq B, \\ 0, & \text{otherwise.} \end{cases}$

(i) Determine the PSD and the power of the noise signal $n_o(t)$. (ii) Given that $K = \frac{3\times 10^{-12}}{8\pi^2}$, what should be the (ideal) filter bandwidth that will give 1 Watt of output power? Aid: $\frac{d}{dt} \leftrightarrow j2\pi f$.

Figure 17: A noise processing system.

16. **Miscellaneous Questions** (30 pts)

(a) Random Process: A random process is described as: $x(t) = A\cos(2\pi t/T)$ where A is a uniform random variable between 0 and 1. Plot 2 distinct (non-zero) sample functions of $x(t)$ for $0 \leq t \leq 2T$ on the same graph. (5 pts)

(b) Measure of Information: Explain briefly why the function $I(x) = -\log(P(x))$ where $P(x)$ is the occurrence probability of event x, can be considered as an effective measure of information. (5 pts)

(c) MAP Criterion: Consider a binary antipodal modulation scheme that uses a basic pulse $g(t)$ with energy 10^{-12}J to communicate through the AWGN channel with PSD of $\frac{N_0}{2} = 5 \times 10^{-7}$ W/Hz. Symbols occur with the following probabilities: $P(1) = 0.8$ and $P(0) = 0.2$. Determine the decision region when maximum aposteori probability (MAP) detector is employed in the receiver. (6 pts)

(d) Bandwidth: A baseband speech signal has a bandwidth of 4 kHz and is sampled at the Nyquist rate, encoded into a PCM format using 8 bits/sample, and then transmitted through the AWGN channel using $M-$ary baseband PAM. Determine the bandwidth required for distortionless transmission when $M = 16$ assuming that the bandwidth of the basic pulse can be approximated by $W = \frac{1}{T}$ Hz where T is the symbol duration in seconds. (6 pts)

(e) Inter Symbol Interference (ISI): Show that $x(t) = \frac{\sin(\frac{\pi t}{T})}{(\frac{\pi t}{T})} \frac{\cos(\frac{\pi t}{2T})}{1-\frac{t^2}{T^2}}$ satisfies the Nyquist criterion for no ISI. Hint: plot $x(t)$ for $t = kT, \quad k = 0, 1, 2, ..$ (8 pts)

17. **M-ary Modulation and Transmission** (20 pts)

(a) The signal constellation for 8-QAM is shown in Fig. 18. If a transmitted signal is $x(t) = s_1(t) + s_8(t - T)$ where T is the symbol duration, sketch $x(t)$ assuming the carrier frequency of $f_c = \frac{1}{T}$. (10 pts)

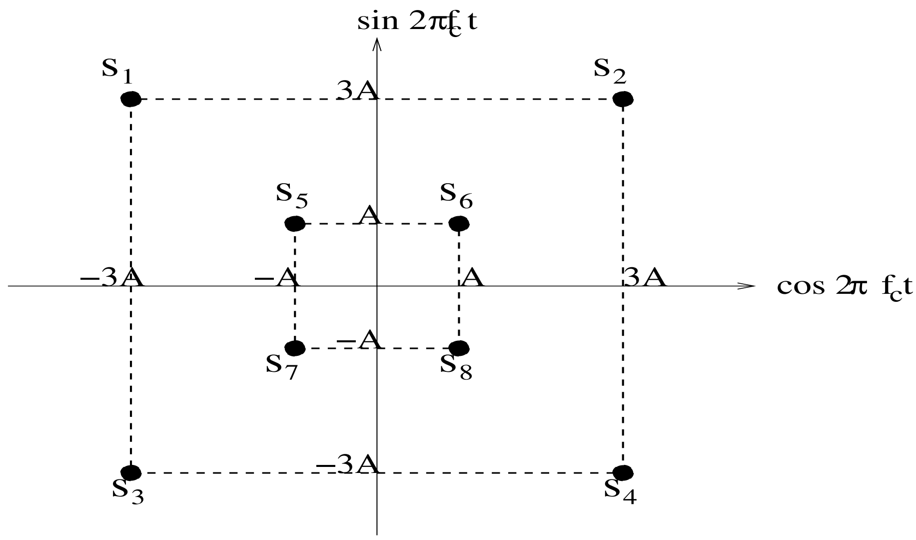


Figure 18: Signal constellation for 8-QAM.

(b) A digital source produces binary data with equal probability. Two types of modulation schemes are considered for transmission of the above information through a AWGN channel.

Scheme A: 0's and 1's are transmitted using $+f(t)$ and $-f(t)$ respectively where

$$f(t) = \begin{cases} A & 0 \le t \le T, \\ 0 & \text{otherwise.} \end{cases}$$

Scheme B: 0's and 1's are transmitted using $g(t)$ and $h(t)$ respectively where

$$g(t) = \begin{cases} B\sin(2\pi t/T) & 0 \le t \le T, \\ 0 & \text{otherwise,} \end{cases} \quad h(t) = \begin{cases} B\cos(2\pi t/T) & 0 \le t \le T, \\ 0 & \text{otherwise.} \end{cases}$$

If both schemes are expected to give *equal* error performance, show that $B = \sqrt{2}A$. Hint: consider constellation points and signal energy. (10 pts)

18. **Maximum Transmission Rate through Channels** (20 pts)

A channel is modelled as a cascaded channel of two binary symmetric channels (BSCs) as shown in Fig. 19.

(a) Find the overall channel capacity of the cascaded connection in terms of p assuming that both channels have the same transition probability as shown in Fig 19. You can also assume that the maximum capacity is achieved when input symbols are equiprobable, i.e., $P(X = 0) =$

$P(X = 1) = \frac{1}{2}$. (10 pts)

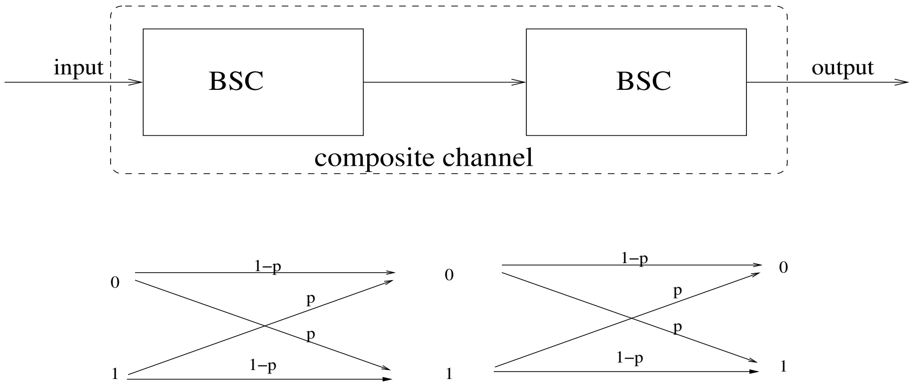


Figure 19: A cascaded system with BSC channels.

(b) It is known apriori that on the average, 60% of the binary source outputs are 1's and the rest are 0's. A baseband PAM scheme is employed with *rectangular pulse* $g(t)$ to transmit through the AWGN channel with noise PSD of $\frac{N_0}{2} = 5 \times 10^{-12}$W/Hz and channel the bandwidth of 1 kHz. As a system designer, your goal is to achieve maximum possible reliable transmission rate of 10^4 bps through the channel. You are constrained to use only the following waveforms for transmission: $f_1(t) = Ag(t)$ and $f_0(t) = 2Ag(t)$ where $f_1(t)$ and $f_0(t)$ are transmitted for 1 and 0 respectively. What should be the value of A? Hint: use Shannon's channel capacity limit. (10 pts)

19. **Autocorrelation and Power Spectral Density** (20 pts)

A random process $n(t)$ has the following characteristics: $E[n(t)] = 0$ and autocorrelation function $\phi_{nn}(\tau) = \alpha^2\delta(\tau)$, where $\delta(.)$ is a delta function and α is a constant. This process is passed through the system as shown in Fig. 20 to generate the output process $y(t)$. The filter response is given by,

$$H(f) = \frac{1}{1 + j2\pi\beta f},$$

where β is a constant.

(a) Show that $x(t)$ a wide sense stationary process? (4 pts)

(b) Determine the power spectral density $\Phi_{xx}(f)$ of the process $x(t)$. (4 pts)

(c) Determine the output power spectral density $\Phi_{yy}(f)$. (4 pts)

(d) Find the autocorrelation function $\phi_{yy}(\tau)$ of $y(t)$. (4 pts)

(e) If the required average output power is $P_{av} = 1$Watt, show that $\beta = \sqrt{\alpha}$. (4 pts)

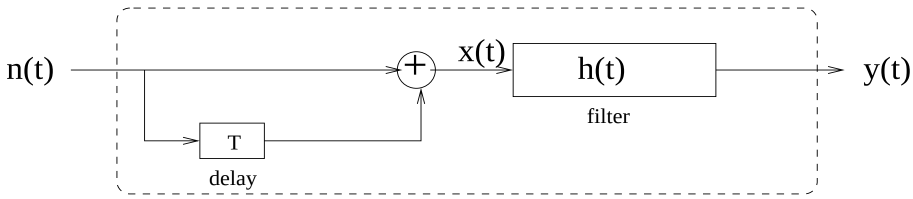


Figure 20: A noise processing system.

20. **Spectrum Shaping** (20 pts)

In a binary digital transmission system, signal $x(t)$ is transmitted where $x(t) = \sum_k b_k g(t - kT)$ with basic pulse $g(t)$ of duration T. The encoding rule is as follows: $b_k = -1$ when the information bit is 0, and $b_k = 2$ when it is 1. It is known apriori that 0's and 1's occur *independently* with probabilities $P(0) = 3/4$ and $P(1) = 1/4$.

(a) Find E(b_k), E(b_k^2) and variance $\sigma_{b_k}^2$ of the sequence b_k. (5 pts)

(b) Find and plot the discrete autocorrelation $\phi_{bb}(m) = \mathrm{E}(b_k b_{k+m})$. (5 pts)

(c) If $g(t)$ has the raised-cosine spectrum with 100% excess bandwidth, sketch $G(f)$. (5 pts)

(d) The power spectral density of $x(t)$ is given as

$$\Phi_{xx}(f) = \frac{\sigma_{b_k}^2}{T}|G(f)|^2 + \frac{[E(b_k)]^2}{T^2} \sum_{m=-\infty}^{\infty} |G(\frac{m}{T})|^2 \delta(f - \frac{m}{T}).$$

For the above raised-cosine pulse sketch $\Phi_{xx}(f)$. (5 pts)

21. **Receiver Design and Performance** (20 pts)

A baseband binary communication system is shown in Fig. 21(a). The information bits are equally likely occurring at the source. Rectangular pulses $g(t)$ are used at the transmitter as shown in Fig. 21(b). The values of $a_k = +1$ and -1 for symbols 1 and 0 respectively.

The noise $n(t)$ is assumed to be zero-mean AWGN with PSD of $N_0/2$. Due to distortion in the channel, the received pulses have different shapes $g^{'}(t)$ as shown in Fig. 21(c). The receiver is unaware of the distortion in the channel and therefore it uses the filter that is matched to the transmitted pulse $g(t)$. *Note that an optimal receiver should have matched to the received pulse* $g^{'}(t)$.

(a) What is the impulse response $h(t)$ of the filter used in the above receiver? Show that $g(t)$ and $g^{'}(t)$ have equal energy; i.e., $E_g = E_{g'} = E_b$. Find E_b. (4 pts)

(b) The one-dimensional decision variable at the output of the sampler is $r = s + n$, where s and n are due to signal and noise respectively.

Given that a "1" was transmitted, find (i) s and (ii) the mean and the variance of r. (4 pts)

(c) Compute and plot $p_R(r|1)$ and $p_R(r|0)$ on the same graph where $p_R(.)$ denotes the probability density function. (3 pts)

(d) Find the probability of error P_e in terms of $Q(.)$ function. (6 pts)

(e) What is the performance loss in dB due to distortion in the channel? (3 pts)

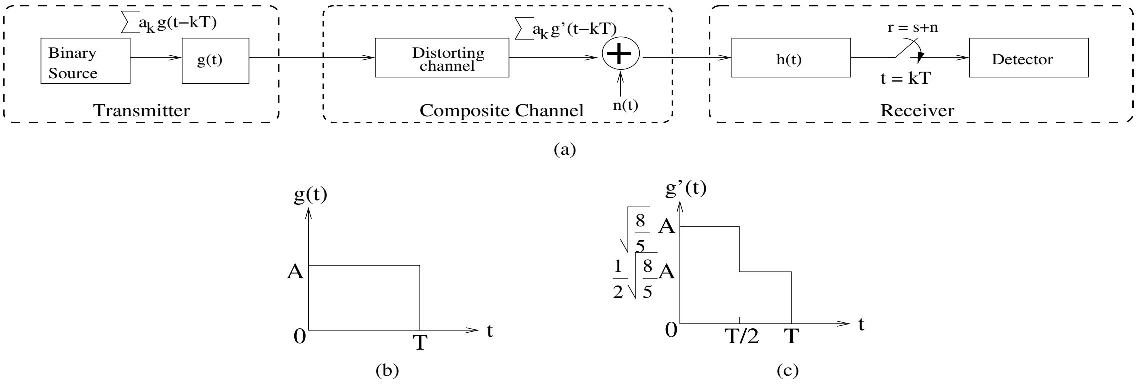


Figure 21: (a) A baseband binary communication system through a channel that introduces distortion, (b) transmitted pulse $g(t)$ and (c) channel distorted pulse $g^{'}(t)$.

22. **Miscellaneous Questions**

(a) [4 pts] The bandwidth of a baseband channel allows a maximum symbol rate of $R_s = 5$ Ksymbols/sec. Since the required bit rate is $R_b = 14$ Kbits/sec, M-ary PAM is decided to be used. Find M.

(b) [6 pts] For the QAM signal constellation shown in Figure 1, specify the Gray code.

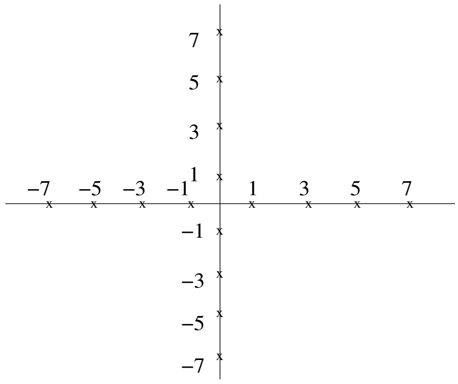


Figure 22: 16-QAM Signal Constellation.

(c) [5 pts] A digital communication system consists of a transmission line with 100 digital repeaters. Binary antipodal signals are used for transmitting the information. If the overall end-to-end error probability is 10^{-6}, determine the probability of error for each repeater and the required E_b/N_0 to achieve this performance in AWGN channel. Assume that each channel between adjacent repeaters has identical error performance. It is given that $Q(5.6) = 10^{-8}$.

(d) [5 pts] In a baseband communication system, the equivalent channel bandwidth is $W = 1500$ KHz. If raised-cosine pulses with 50% excess bandwidth are used, determine the maximum symbol rate that would not cause any ISI.

23. **Design of Variable-length Codes and Encoding Efficiency**

A 4-level nonuniform quantizer for a Gaussian-distributed signal amplitude results in the four levels, a_1, a_2, a_3 and a_4, with corresponding probabilities of occurrence, $p_1 = p_2 = 1/3$ and $p_3 = p_4 = 1/6$. Let J be the number of symbols in a block.

(a) [3 pts] Determine the entropy of the source.

(b) [4 pts] Design a Huffman code that encodes a single level ($J = 1$) at a time and determine the average bit rate.

(c) [6 pts] Design a Huffman code that encodes two output levels ($J = 2$) at a time and determine the average bit rate.

(d) [3 pts] Determine the efficiency of the encoding scheme in (b) and (c). From the efficiency, what can you conclude about the encoding of blocks ?

(e) [4 pts] What is the lower bound on the bit rate obtained by encoding J output levels as $J \to \infty$?

24. **Power Spectral Density and Shaping**

In a digital signaling scheme, $x(t) = \sum_k b_k g(t-kT)$, where $g(t)$ is the basic pulse with pulse duration of T. The encoding rule is as follows: $b_k = -\frac{A}{3}$ when the information bit is 0, and $b_k = \frac{2A}{3}$ when it is 1. It is known a priori that 0's and 1's are independent with probabilities $P(0) = 3/4$ and $P(1) = 1/4$.

(a) [3 pts] Find $\mathrm{E}(b_k)$, $\mathrm{E}(b_k^2)$ and variance, σ_b^2, of the sequence b_k where $\mathrm{E}(.)$ denotes expectation.

(b) [7 pts] Plot the discrete autocorrelation function, $\phi_{bb}(m) = \mathrm{E}(b_k b_{k+m})$ for A = 12.

(c) [3 pts] If $g(t)$ is a raised-cosine pulse with 50% excess bandwidth, sketch $G(f)$. Hint: See page 561 in the text book or page 7 on the handouts of Chapter 9.

(d) [7 pts] The power spectral density expression is given as

$$\Phi_{xx}(f) = \frac{\sigma_b^2}{T}|G(f)|^2 + \frac{(E(b_k))^2}{T^2}\sum_{m=-\infty}^{\infty}|G(\frac{m}{T})|^2\delta(f-\frac{m}{T}).$$

For the above raised-cosine pulse and with $A = 12$, sketch $\Phi_{xx}(f)$.

25. **Optimum Receiver Design in AWGN**

A binary digital communication system employs the signals $s(t) = \{s_0(t) \quad s_1(t)\}$ where,

$$s_0(t) = 0, \qquad 0 \le t \le T,$$

$$s_1(t) = A, \qquad 0 \le t \le T,$$

for transmitting the information. The demodualator cross-correlates the received signal $r(t)$ with $s(t)$ and samples the output of the correlator $t+T$.

(a) [8 pts] Determine (i) the optimum detector for an AWGN channel and (ii) the optimum threshold assuming that the signals are equiprobable.

(b) [8 pts] Given that The noise power spectral density is $N_0/2$, determine the probability of symbol error as a function of SNR.

(c) [4 pts] How does this signaling scheme compare with the antipodal signaling ? Hint: Compare the energy required to achieve the same error performance.

26. **Signal Space Representation and Minimum Distance Detection**

Three messages m_1, m_2 and m_3 are to be transmitted over an AWGN channel with noise power spectral density $N_0/2$. The messages are

$$s_1(t) = \begin{cases} A, & 0 \le t \le T, \\ 0, & \text{otherwise}, \end{cases}$$

$$s_2(t) = -s_3(t) = \begin{cases} A, & 0 \le t \le T/2, \\ -A, & T/2 < t \le T, \\ 0, & \text{otherwise}, \end{cases}$$

where A is a constant.

(a) [2 pts] What is the dimensionality of the signal space ?

(b) [3 pts] Find a basis for the signal space. Hint: Can be done by inspection.

(c) [4 pts] Give a vector representation of the signals and draw the signal constellation for this problem.

(d) [3 pts] Assuming equiprobable signals, sketch the optimal decision regions for $m_i, i = 1, 2, 3$.

(e) [8 pts] Prove that the message m_1 is more vulnerable to errors. Hint: Consider $P(\text{error}|m_i \text{ transmitted}), i = 1, 2, 3$.

27. **Performance of Baseband 4-ary Signaling**

A set of transmitted signals in a baseband 4-ary scheme is shown in Figure 2. All the symbols are equally-likely.

(a) [3 pts] Find the set of basis function(s), $\{f_i\}$, needed to represent the above 4 signals.

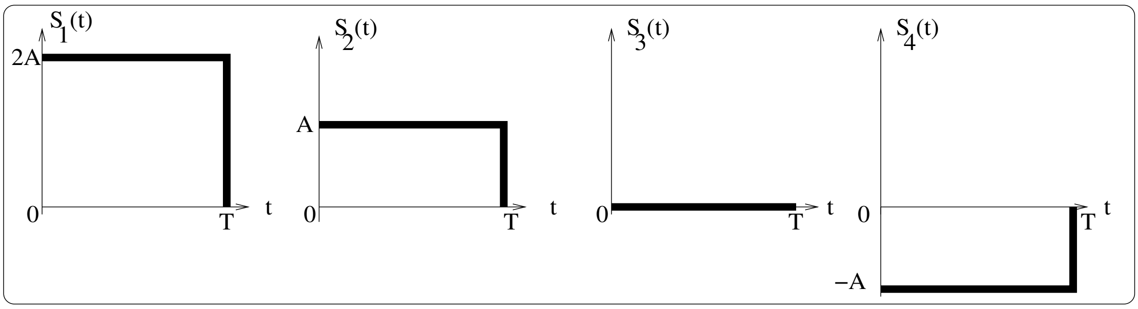


Figure 23: Transmitted Signals in a 4-ary Signaling Scheme.

The transmission is through a wireless channel with impulse response is $h_c(t) = \delta(t)$, where $\delta(t)$ denotes the delta function. Noise is AWGN denoted by $n(t)$ with zero-mean and power spectral density of $N_0/2$. The received signal $x(t)$ is passed through a filter as shown in Figure 3. In the figure B denotes a constant. The decision variable shown in the figure has a component m_i due to the transmitted signal and a component n due to the noise. Function, $f_Y(y)$ denotes the probability density function of sampled output, y.

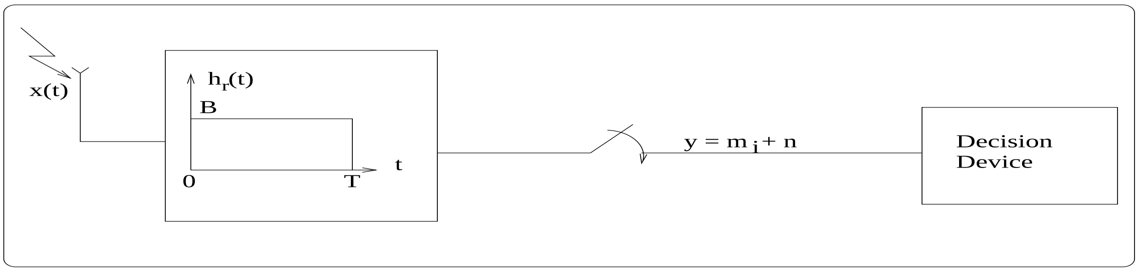


Figure 24: Receiver Structure in AWGN Channel.

(b) [7 pts] Show (quantitatively) that the matched-filter shown in Figure 3 is optimal in the sense it minimizes the probability of error. Hint:

First determine $x(t)$ and then consider minimum-distance criterion.

(c) [4 pts] Sketch $f_Y(y|1), f_Y(y|2), f_Y(y|3)$ and $f_Y(y|4)$, on the same figure. Here, $f_Y(y|i)$ denotes the conditional probability for i th signal transmitted.

(d) [6 pts] Find the probability of symbol error. Hint: Consider for example,

P(error|1st signal transmitted) and its relationship to the overall error probability.

28. **Probability and Random Process:** (15 pts)

Determine the mean and the autocorrelation function of the random process

$$X(t) = A \cos\,(\omega_c t + \theta),$$

where θ is an random variable uniformly distributed in the range $(0, 2\pi)$ and A is a constant. Is the process wide-sense stationary ?

29. **Signal and Vector Space:** (15 pts)

Consider the three waveforms $f_n(t)$ shown in Fig. 1. Are these waveforms are orthogonal ?

30. **Autocorrelation and Power Spectral Density:** (25 pts)

The white noise process $X(t)$ is the input to the channel whose transfer function is given by,

$$H(f) = \frac{1}{1 + j2\pi\varrho f},$$

where ϱ is a constant. The input has the following characteristics: $E[X(t)] = 0$ and autocorrelation function $\phi_{xx}(\tau) = \sigma^2\delta(\tau)$, where $\delta(.)$ is a delta function. The corresponding output process is $Y(t)$. Determine the following:

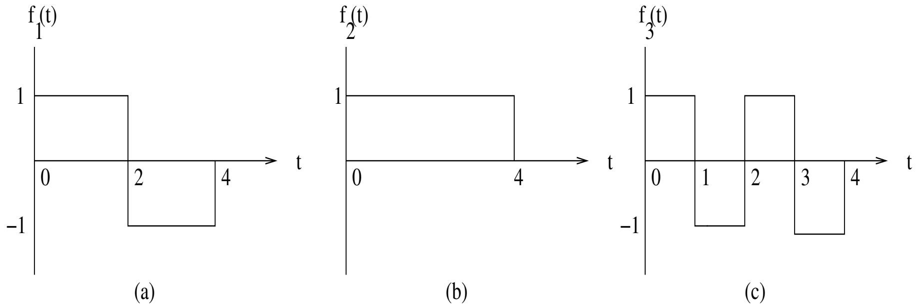


Figure 25: Waveforms, $f_n(t), n = 1, 2, 3$.

(a) the output power spectral density $\Phi_{yy}(f)$?

(b) the autocorrelation function of the output process $\phi_{yy}(\tau)$?

(c) the average output power ?

31. **Source Entropy and Block Coding:** (25 pts)

A discrete memoryless channel has an alphabet of four letters, $x_i, i = 1, 2, 3, 4$, each occurring with probability $\frac{1}{3}, \frac{1}{3}, \frac{1}{6}$ and $\frac{1}{6}$ respectively. Evaluate the efficiency of a fixed-length binary code in which,

(a) each letter is encoded separately into binary sequence ?

(b) two letters are encoded separately into binary sequence ?

(c) from the above results in (a) and (b), what can you conclude about the coding of symbols ?

32. **Noisy Channel Capacity:** (20 pts)

A communication channel is characterized as a band-limited (to 3 KHz) AWGN waveform channel with noise power spectral density of $N_0 = 8 \times 10^{-8}$ Watts/Hz. The average input power to the channel is 24 mWatts.

(a) determine the capacity of the channel ?

(b) is it possible to transmit reliably a signal that has a maximum frequency component at 5 KHz and sampled ar Nyquist rate and encoded with 8 bits/sample? Justify your answer.