Exponential Distribution: Basic Facts

- Density

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases} \qquad \lambda > 0$$

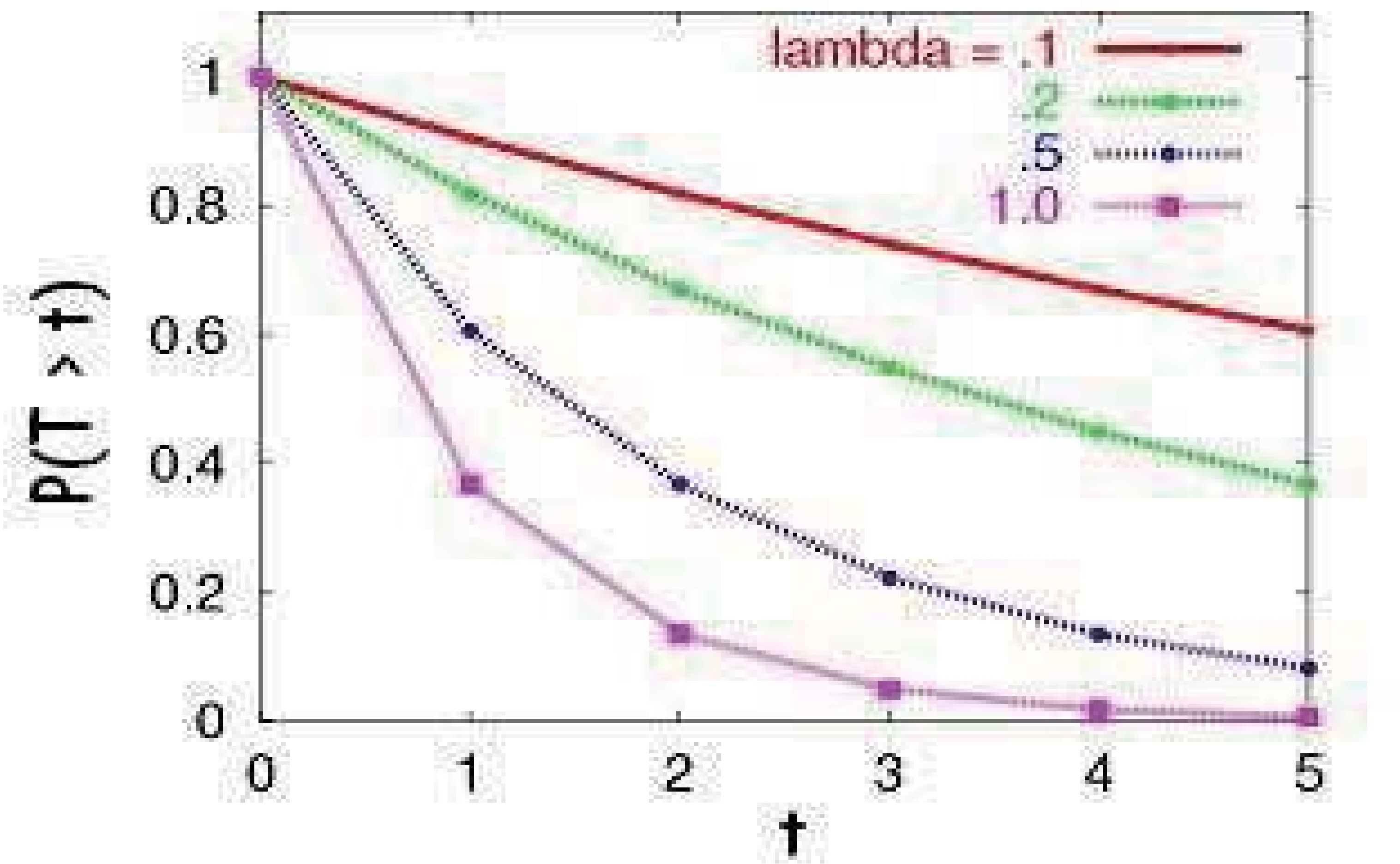
- CDF

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

- CF

$$\phi(\omega) = E[e^{j\omega X}] = \frac{\lambda}{\lambda - j\omega}$$

- Mean $E[X] = 1/\lambda$
- Variance $Var[X] = 1/\lambda^2$


lambda = .1
.2
.5
1.0
P(T > t)
1
0.8
0.6
0.4
0.2
0
0
1
2
3
4
5
t

Key Property: *Memorylessness*

$$P\{X > s + t | X > t\} = P\{X > s\} \quad \text{for all } s, t \geq 0$$

- Reliability: Amount of time a component has been in service has no effect on the amount of time until it fails
- Inter-event times: Amount of time since the last event contains no information about the amount of time until the next event
- Service times: Amount of remaining service time is independent of the amount of service time elapsed so far.
- An example of a memoryless RV, T
  - Let T be the time of arrival of a memoryless event, E
  - Choose any constant, D
  - $P(T > D) = P(T > x + D | T > x)$ for any x
  - We "checked" to see if E occurred before x and found out that it didn't.
  - Given that it did not occur before time x, the likelihood that it won't occur by time $D + x$ is the same as that the timer is reset to 0 and it won't occur by time D.

Other Useful Properties

- Competing Exponentials: If X_1 and X_2 are independent exponential RVs with parameters λ_1 and λ_2, respectively, then

$$P\{X_1 < X_2\} = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

Proof: X_1 and X_2 are independent exponential random variables with λ_1 and λ_2.

$$\begin{aligned} P(X_1 < X_2) &= \int_0^\infty P(X_1 < X_2 | X_2 = t)\, P(X_2 = t) dt \\ &= \int_0^\infty P(X_1 < t) P(X_2 = t) dt \\ &= \int_0^\infty (1 - e^{-\lambda_1 t}) \lambda_2 e^{-\lambda_2 t} dt \;= \frac{\lambda_1}{\lambda_1 + \lambda_2} \end{aligned}$$

- Minimum of exponentials: If $X_1, X_2, \cdots, X_n$ are independent exponential RVs where X_i has parameter λ_i, then $\min(X_1, X_2, \cdots, X_n)$ is exponential with parameter $\lambda_1 + \lambda_2 + \cdots + \lambda_n$.

Proof: Define random variable Y=min$\{X_1, X_2, \ldots, X_n\}$

$X_1, \cdots, X_n$ are independent exponential r.v.s with λ_i

$$\begin{aligned} Pr(Y > x) &= Pr\{\min\{X_1, \cdots, X_n\} > x\} = Pr\{X_1 > x, \cdots, X_n > x\} \\ &= Pr\{X_1 > x\} \cdots Pr\{X_n > x\} = \int_x^\infty \lambda_1 e^{-\lambda_1 z} dz \cdots \int_x^\infty \lambda_n e^{-\lambda_n z} dz \\ &= e^{-\lambda_1 x} \cdots e^{-\lambda_n x} = e^{-(\lambda_1 + \cdots + \lambda_n)x} \end{aligned}$$

it is equivalent to an exponential r.v. with $\lambda = \lambda_1 + \cdots + \lambda_n$.

$$F_Y(y) = Pr(Y \le y) = 1 - Pr(Y > y) = 1 - e^{-\lambda y}$$

Counting Process

A stochastic process $\{N(t), t \geq 0\}$ is a *counting process* if $N(t)$ represents the total number of events that have occurred in $(0, t]$. Then $\{N(t), t \geq 0\}$ must satisfy:

- $N(t) \geq 0$.
- $N(t)$ is an integer for all t.
- If $s < t$, then $N(s) \leq N(t)$
- For $s < t$, $N(t) - N(s)$ is the number of events that occur in the interval $(s, t]$.

Stationary and Independent Increments

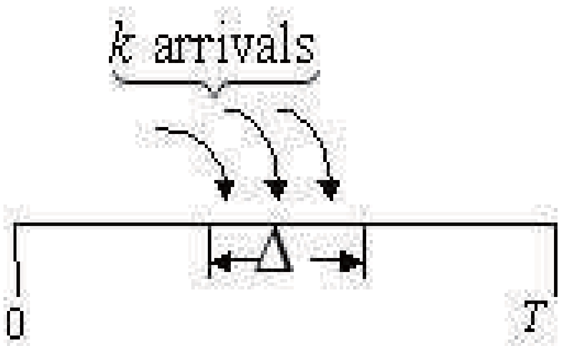
- A counting process has independent increments if, for any $0 \leq s < t \leq u < v$, $N(t) - N(s)$ is independent of $N(v) - N(u)$. That is, the numbers of events that occur in non-overlapping intervals are independent random variables.
- A counting process has *stationary increments* if the distribution if, for any $s < t$, the distribution of $(N(t) - N(s))$ depends only on the length of the time interval, $(t - s)$

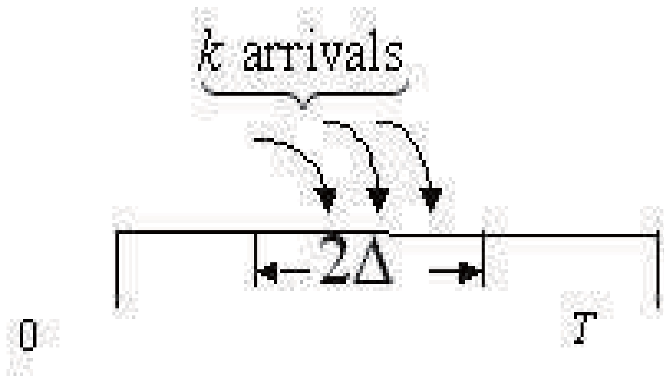
Poisson Process Definition

A counting process $\{N(t), t \geq 0\}$ is a Poisson process with rate λ, $\lambda > 0$, if

- $N(0) = 0$.
- The process has independent increments
- The number of events in any interval of length t follows a Poisson distribution with mean λt (therefore, it has stationary increments), i.e.,

$$P\{N(t+s) - N(s) = n\} = \frac{e^{-\lambda t}(\lambda t)^n}{n!} \quad n = 0, 1, \cdots$$





$$P\{\text{“}k \text{ arrivals occur in an interval of duration } \Delta\text{”}\} = e^{-\lambda_1}\frac{\lambda_1^k}{k!}$$

where

$$\lambda_1 = \lambda \cdot \Delta$$

It follows that

$$P\{\text{“}k \text{ arrivals occur in an interval of duration } 2\Delta\text{”}\} = e^{-\lambda_2}\frac{\lambda_2^k}{k!}$$

since in that case

$$\lambda_2 = \lambda \cdot 2\Delta = 2\lambda_1$$

Poisson arrivals over an interval form a Poisson random variable whose parameter depends on the duration of that interval.

Example: Data packets transmitted by a modem over a phone line form a Poisson process of rate 10 packet/sec. Using M_k to denote the number of packets transmitted in the kth hour, find the joint pmf of M_1 and M_2.

The first and second hours are nonoverlapping intervals. Since one hour equals 3600 seconds and the Poisson process has a rate of 10 packet/sec, the expected number of packets in each hour is $E[M_i] = \lambda t = \alpha = 36,000$. This implies M_1 and M_2 are independent Poisson random variable each with pmf

$$p_{M_i}(m) = \begin{cases} \frac{\alpha^m e^{-\alpha}}{m!} & m = 0, 1, 2, \cdots \\ 0 & o.w. \end{cases}$$

Since M_1 and M_2 are independent, the joint pmf is

$$p_{M_1,M_2}(m_1, m_2) = p_{M_1}(m_1)p_{M_2}(m_2) = \begin{cases} \frac{\alpha^{m_1+m_2} e^{-2\alpha}}{m_1!m_2!} & m_1 = 0, 1, 2, \cdots ; m_2 = 0, 1, 2, \cdots \\ 0 & o.w. \end{cases}$$

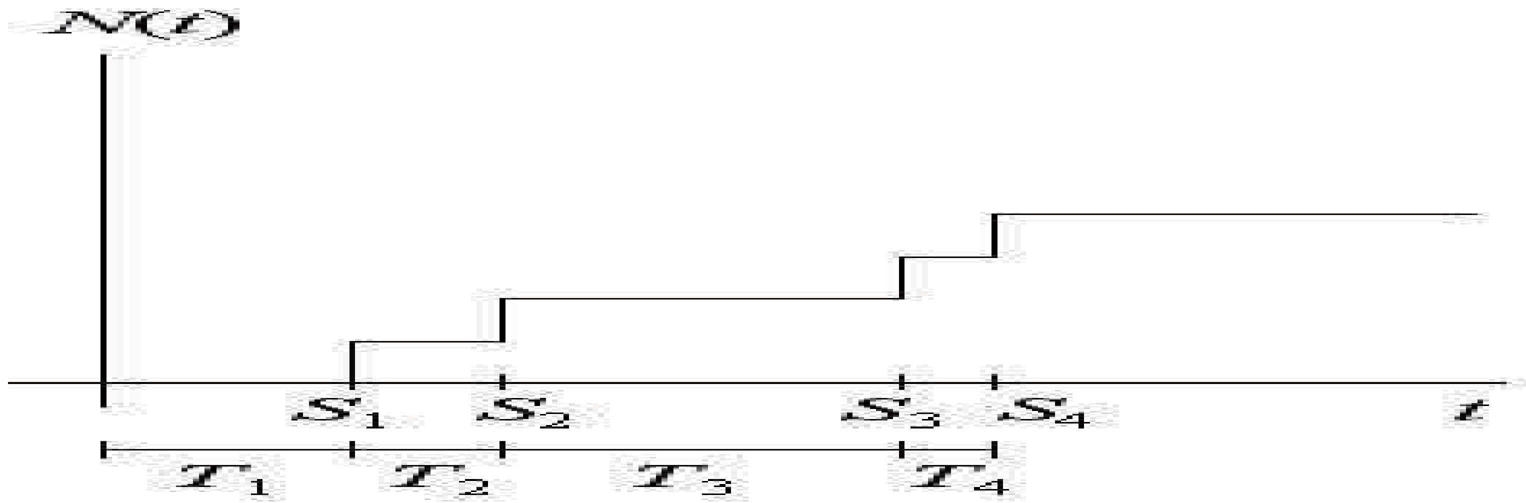
Interarrival and Waiting Times

The times between arrivals $T_1, T_2, \cdots$ are independent exponential r.v.s with mean $1/\lambda$:

$$P(T_1 > t) = P(N(t) = 0) = e^{-\lambda t}$$

The (total) waiting time until the nth event has a gamma distribution:

$$S_n = \sum_{i=1}^{n} T_i$$



Inter-arrival Distribution for Poisson Processes

Let T_1 denote the time interval (delay) to the first arrival from any fixed point t_0. To determine the probability distribution of the random variable T_1, we argue as follows: Observe that the event $T_1 > t$ is the same as "$N(t_0, t_0 + t) = 0$", or the complement event $T_1 \le t$ is the same as the event "$N(t_0, t_0 + t) > 0$" . Hence the distribution function of T_1 is given by

$$F_{T_1}(t) = P(T_1 \le t) = P(N(t) > 0) = P[N(t_0, t_0 + t) > 0] = 1 - e^{-\lambda t}$$

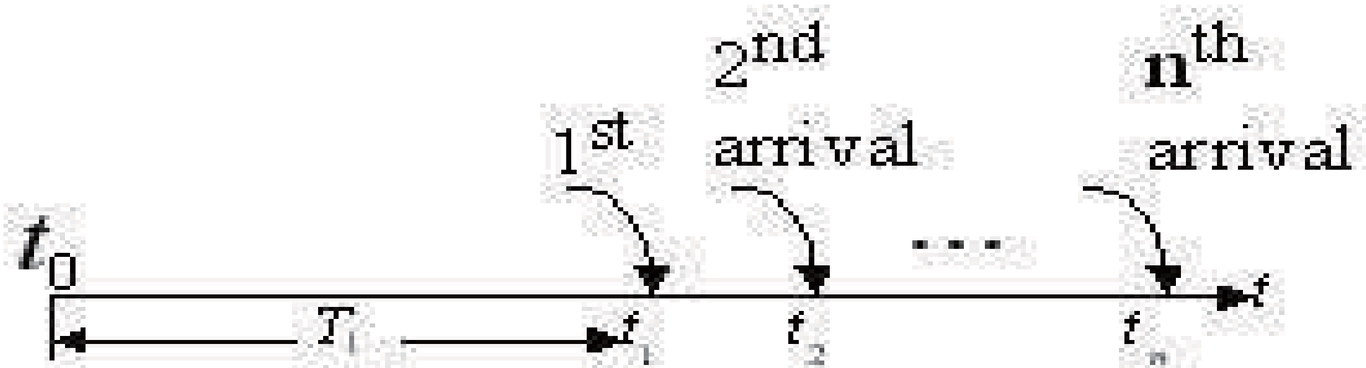


Figure 1: Inter-arrival time.

and hence its derivative gives the density function for T_1 to be

$$f_{T_1}(t) = \lambda\, e^{-\lambda t}, \qquad t \geq 0 \tag{4}$$

i.e., T_1 is an exponential random variable with parameter λ so that $E(T_1) = 1/\lambda$.

Similarly, let S_n represent the nth random arrival point for a Poisson process. Then

$$F_{S_n}(t) = P(S_n \leq t) = P(N(t) \geq n) = 1 - P[N(t) < n] = 1 - \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \tag{5}$$

and hence

$$f_{S_n}(x) = -\sum_{k=1}^{n-1} \frac{\lambda(\lambda x)^{k-1}}{(k-1)!}\, e^{-\lambda x} + \sum_{k=0}^{n-1} \frac{\lambda(\lambda x)^k}{k!}\, e^{-\lambda x} = \frac{\lambda^n x^{n-1}}{(n-1)!}\, e^{-\lambda x}, \quad x \geq 0 \tag{6}$$

which represents a gamma density function. i.e., the waiting time to the nth Poisson arrival instant has a gamma distribution. Moreover

$$S_n = \sum_{i=1}^{n} T_i$$

where T_i is the random inter-arrival duration between the $(i-1)$th and ith events. Notice that T_i's are independent, identically distributed (iid) exponential RVs with mean $1/\lambda$.

Classified Poisson Process

- $\{N(t); t \geq 0\}$ is Poisson process with rate λ
- Classified each arrival as type I or type II, according to P(type I)=p, P(type II)=$1-p$
- Let
  - $N_1(t)$ = number of type I arrivals in $(0, t]$
  - $N_2(t)$ = number of type II arrivals in $(0, t]$
  - and $N(t) = N_1(t) + N_2(t)$
- Then

$$Pr\{N_1(t)=n, N_2(t)=m\} = \sum_{k=0}^{\infty} Pr\{N_1(t)=n, N_2(t)=m|N(t)=k\} \cdot Pr\{N(t)=$$

$$= \quad Pr\{N_1(t)=n, N_2(t)=m|N(t)=n+m\}Pr\{N(t)=n+m\}$$

$$= \quad \begin{pmatrix} n+m \\ n \end{pmatrix} p^n(1-p)^m \frac{e^{-\lambda t}(\lambda t)^{n+m}}{(n+m)!}$$

$$= \quad \frac{p^n(1-p)^m e^{-\lambda t}(\lambda t)^n(\lambda t)^m}{n!\, m!} = \frac{e^{-\lambda t p}(\lambda t p)^n}{n!} \frac{e^{-\lambda t(1-p)}(\lambda t(1-p))^m}{m!}$$

Therefore,

$$Pr\{N_1(t) = n\} = \sum_{m=0}^{\infty} Pr\{N_1(t) = n, N_2(t) = m\}$$

$$= \frac{e^{-\lambda tp}(\lambda tp)^n}{n!} \underbrace{\sum_{m=0}^{\infty} \frac{e^{-\lambda t(1-p)}(\lambda t(1-p))^m}{m!}}_{=1,\text{sum over pmf}}$$

$$= \frac{e^{-\lambda tp}(\lambda tp)^n}{n!} \sim \text{Poisson r.v. with parameter} \lambda p$$

- Similarly, Pr$\{N_2(t) = m\} \sim$ Poisson with parameter $\lambda(1-p)$.
- Can be extended to K classes (by induction).
- Let green packets arrive as a Poisson process with rate λ_1, and red packets arrive as a Poisson process with rate λ_2, then green+red packets arrive as a Poisson process with rate $\lambda_1 + \lambda_2$, and

$$Pr\{\text{next packet is red}\} = \lambda_2/(\lambda_1 + \lambda_2)$$

Racing Poisson Process

- Two independent Poisson process, N_A and N_B,
  - $\{N_A(t); t \geq 0\} \sim$ Poisson withλ_A, $\{N_B(t); t \geq 0\} \sim$ Poisson withλ_B
  - S_A^n time in process N_A, S_B^n time in process N_B,
- $S_A^1 \sim$ exponential with rate λ_A, and $S_B^1 \sim$ exponential with rate λ_B, independent with each other.

$$Pr\{S_A^1 < S_B^1\} = \frac{\lambda_A}{\lambda_A + \lambda_B} \tag{8}$$

- then we have

$$\begin{aligned} & Pr\{S_A^2 < S_B^1\} \\ = \; & Pr\{S_A^2 < S_B^1 | S_A^1 < S_B^1\} \cdot Pr\{S_A^1 < S_B^1\} + Pr\{S_A^2 < S_B^1 | S_A^1 > S_B^1\} \cdot Pr\{S_A^1 > S_B^1\} \\ = \; & Pr\{S_A^1 < S_B^1\} \cdot Pr\{S_A^1 < S_B^1\} \\ = \; & \left(\frac{\lambda_A}{\lambda_A + \lambda_B} \right)^2 \end{aligned} \tag{9}$$