EE8103 Assignment 4

1. A random variable T is picked according to a uniform distribution on $(0, 1]$. Then, another random variable U is picked according to a uniform distribution on $(0, T]$. Find the marginal CDF $F_U(u)$ for the random variable U.

2. At 12 noon on a weekday, we begin recording new call attempts at a telephone switch. Let X denote the arrival time of the first call, as measured by the number of seconds after noon. Let Y denote the arrival time of the second call. In the most common model used in the telephone industry, X and Y are continuous random variable with joint pdf

$$f_{X,Y}(x, y) = \begin{cases} \lambda^2 e^{-\lambda y} & 0 \le x < y \\ 0 & \text{o.w.} \end{cases}$$

where λ is a positive constant. Find the marginal pdf $f_X(x)$ and $f_Y(y)$ and the conditional pdf $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$

3. X is a Poisson random variable with parameter λ and Y is a normal random variable with mean μ and variance σ^2. Further X and Y are given to be independent.

   (a) Find the joint characteristic function of X and Y.

   (b) Define $Z = X + Y$. Find the characteristic function of Z.

4. The joint probability mass function of X and Y, $p(x, y)$ is given by

$$\begin{array}{lll} p(1,1) = \frac{1}{9} & p(2,1) = \frac{1}{3} & p(3,1) = \frac{1}{9} \\ p(1,2) = \frac{1}{9} & p(2,2) = 0 & p(3,2) = \frac{1}{18} \\ p(1,3) = 0 & p(2,3) = \frac{1}{6} & p(3,3) = \frac{1}{9} \end{array}$$

(a) Computer $E[X|Y = i]$ for $i = 1, 2, 3$.

(b) Are the random variables X and Y independent?

5. The joint density function of X and Y is given by

$$f(x, y) = \frac{e^{-y}}{y}, \qquad 0 < x < y, \quad 0 < y < \infty$$

compute $E[X^2|Y = y]$.

6. Consider the example which refers to a miner trapped in a mine. Let N denote the total number of doors selected before the miner reaches safety. Also, let T_i denote the travel time corresponding to the ith choice, $i \leq 1$. Again, let X denote the time when the miner reaches safety. **(refer to Example 20 Chapter 3 in lecture slides)**

(a) Given an identity that relates X to N and the T_i.

(b) What is $E[N]$?

(c) What is $E[T_N]$?

(d) What is $E[\sum_{i=1}^{N} T_i|N = n]$?

(e) Using the preceding, what is $E[X]$?

~~7.~~ Consider a Markov chain with state space $\{0 , 1, 2, 3, 4\}$ and transition matrix

Question 7 is deleted.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1/4 & 3/4 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 0 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{bmatrix} \qquad (1)$$

(a) Classify the states of the chain.

(b) Determine the stationary distribution for states 1 and 2.

(c) For the transient states, calculate s_{ij}, the expected number of visits to transient state j, given that the process started in transient state i.

(d) Find Prob$(X_5 = 2|X_3 = 1)$.

8. Trials are performed in sequence. If the last two trials were successes, then the next trial is a success with probability 0.8; otherwise the next trial is a success with probability 0.5. In the long run, what proportion of trials are successes?

9. For a series of dependent trials the probability of success on any trial is $(k+1)/(k+2)$ where k is equal to the number of successes on the previous two trials. Compute $\lim_{n=\infty} P\{\text{success on the } n\text{th trial}\}$.

10. Consider a Markov chain with states 0,1,2,3,4. Suppose $P_{0,4} = 1$; and suppose that when the chain is in state i, $i > 0$, the next state is equally likely to be any of the states 0,1,...,$i-1$. Find the limiting probabilities of this Markov chain.

11. (Textbook 4.1.6)
Can the following function be the joint CDF of random variables X and Y?

$$F(x, y) = \begin{cases} 1 - e^{-(x+y)} & x \geq 0, y \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

12. (Textbook 4.3.5)
Random variables N and K have the joint PMF

$$P_{N,K}(n, k) = \begin{cases} \frac{100^n e^{-100}}{(n+1)!} & k = 0, 1, \ldots, n; \\ & n = 0, 1, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Find the marginal PMF $P_N(n)$. Show that the marginal PMF $P_K(k)$ satisfies $P_K(k) = P[N > k]/100$.

13. (Textbook 4.4.2)
Random variables X and Y have joint PDF

$$f_{X,Y}(x, y) = \begin{cases} cxy^2 & 0 \leq x \leq 1, 0 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the constant c.

(b) Find $P[X > Y]$ and $P[Y < X^2]$.

(c) Find $P[\min(X, Y) \leq 1/2]$.

(d) Find $P[\max(X, Y) \leq 3/4]$.

14. (Textbook 4.6.6)

Random variables X and Y have joint PDF

$$f_{X,Y}(x, y) = \begin{cases} x + y & 0 \leq x \leq 1, 0 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $W = \max(X, Y)$.

(a) What is S_W, the range of W?

(b) Find $F_W(w)$ and $f_W(w)$.

15. (Textbook 4.8.6)

Random variables X and Y have joint PDF

$$f_{X,Y}(x, y) = \begin{cases} (4x + 2y)/3 & 0 \leq x \leq 1; \\ & 0 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $A = \{Y \leq 1/2\}$.

(a) What is $P[A]$?

(b) Find $f_{X,Y|A}(x, y)$, $f_{X|A}(x)$, and $f_{Y|A}(y)$.

16. (Textbook 4.9.4)

Random variables X and Y have joint PDF

$$f_{X,Y}(x, y) = \begin{cases} 2 & 0 \leq y \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find the PDF $f_Y(y)$, the conditional PDF $f_{X|Y}(x|y)$, and the conditional expected value $E[X|Y = y]$.

17. (Textbook 4.9.9)

Random variables N and K have the joint PMF

$$P_{N,K}(n, k) = \begin{cases} \frac{100^n e^{-100}}{(n+1)!} & k = 0, 1, \ldots, n; \\ & n = 0, 1, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Find the marginal PMF $P_N(n)$, the conditional PMF $P_{K|N}(k|n)$, and the conditional expected value $E[K|N = n]$. Express the random variable $E[K|N]$ as a function of N and use the iterated expectation to find $E[K]$.