EE8103 Assignment 3

1. A random variable X undergoes the transformation $Y = a/X$, where a is a real number. Find the density function of Y.

2. A random variable X is uniformly distributed on the interval $(-a, a)$. It is transformed to a new variable Y by the transformation $Y = cX^2$. Find and sketch the density function of Y.

3. A Gaussian voltage random variable X has a mean of $\mu = 0$, and variance of $\sigma^2 = 9$. The voltage X is applied to a square-law, full-wave diode detector with a transfer characteristic $Y = 5X^2$. Find the mean value of the output voltage Y.

4. Consider a probability space (Ω, F, P). Let $\Omega = \{\xi_1, ..., \xi_5\} = \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$ with $P(\xi_i) = \frac{1}{5}$, $i = 1, ..., 5$. Define two r.v.'s as follows:

$$X(\xi) = \xi \qquad\qquad Y(\xi) = \xi^2 \tag{1}$$

   (a) Show that X and Y are dependent r.v.'s.

   (b) Show that X and Y are uncorrelated.

5. Consider the recursion known as a *first-order moving average* given by

$$X_n = Z_n - aZ_{n-1} \qquad\qquad |a| < 1 \tag{2}$$

   where X_n, Z_n, Z_{n-1} are all r.v.'s for $n = .., -1, 0, 1, ...$ Assume $E[Z_n] = 0$ for all n; $E[Z_n Z_j] = 0$ for all $n \neq j$; and $E[Z_n^2] = \sigma^2$ for all n. Compute $R_n(k) = E[X_n X_{n-k}]$ for $k = 0, \pm 1, \pm 2, ...$

6. The joint probability density function of random variables X and Y is

$$f_{X,Y}(x, y) = \begin{cases} cxy & 0 \leq x \leq 1, 0 \leq y \leq 2 \\ 0 & \text{o.w.} \end{cases}$$

   What is the value of the constant c? What is the probability of the event A that $X^2 + Y^2 \leq 1$?

7. A point is uniformly distributed within the disk of radius 1. That is, its density is

$$f(x, y) = C, \qquad 0 \leq x^2 + y^2 \leq 1$$

   Find the probability that its distance from the origin is less than x, $0 \leq x \leq 1$.

8. Suppose that X and Y are independent continuous random variables. Show that

$$P(X \leq Y) = \int_{-\infty}^{\infty} F_X(y) f_Y(y) dy$$

.

9. Let X and Y be independent random variables with means μ_X and μ_Y and variances σ_X^2 and σ_Y^2. Show that

$$Var(XY) = \sigma_X^2 \sigma_Y^2 + \mu_Y^2 \sigma_X^2 + \mu_X^2 \sigma_Y^2.$$

10. Let X and Y be independent normal random variables, each having parameters μ and σ^2. Show that $X + Y$ is independent of $X - Y$.

    Hints: Find their joint moment generating function.

11. Let $\phi(\omega_1, \omega_2, ..., \omega_n)$ denote the joint moment characteristic function of random variables $X_1, X_2, ..., X_n$.

    (a) Explain how the characteristic function of X_i, $\phi_{X_i}(\omega_i)$, can be obtained from $\phi(\omega_1, \omega_2, ..., \omega_n)$.

    (b) Show that $X_1, X_2, ..., X_n$ are independent if and only if

    $$\phi(\omega_1, \omega_2, ..., \omega_n) = \prod_{i=1}^{n} \phi_{X_i}(\omega_i)$$

    Note: You can interprete $\phi(\omega)$ as either characteristic function or moment generating function.

12. Let the RV X have a uniform distribution in the interval $(9, 11)$. Define the RV $Y = 9/X$, and find the distribution of Y. The example is of a resistor of nominal value 10 Ω, with tolerance of $\pm 10\%$ (its value is represented by X) and Y is the current in the resistor when a 9-volt battery is applied at its terminals.

13. Considering the case of a saturation amplifier operating on a Gaussian RV. We assume that the input X has a Gaussian density function, with $m = 0$ and $\sigma = 0.5$. The amplifier, who output is assumed to be Y, follows:

    $$Y = g(X) = \begin{cases} -a & X < -1 \\ aX & -1 < X < 1 \\ +a & 1 < X \end{cases}$$

    Find the distribution and density functions of Y.

14. Let X and Y be independent RVs with Cauchy distributions having parameters α and β, respectively. Use the characteristic function to find the density of their sum Z.