Assignment 5 Solution

1. **Solution for Q1:**

   X be an exponential RV with rate λ,

$$f_X(x) = \lambda e^{-\lambda x} \quad x \geq 0$$

(a) we have

$$E[X|X < c] = \frac{\int_0^c x \cdot f_X(x)\,dx}{\int_0^c \cdot f_X(x)\,dx} = \frac{A}{B}$$

where

$$\begin{aligned} A &= \int_0^c x \cdot \lambda e^{-\lambda x}\,dx = -\int_0^c x\,d\,e^{-\lambda x} \\ &= xe^{-\lambda x}|_0^c + \int_0^c e^{-\lambda x}\,dx = \frac{1}{\lambda} - \left(c + \frac{1}{\lambda}\right) e^{-\lambda c} = \frac{1}{\lambda}\left(1 - e^{-\lambda c}\right) - c \cdot e^{-\lambda c} \end{aligned}$$

and

$$B = \int_0^c f_X(x)\,dx = \int_0^c \lambda e^{-\lambda x}\,dx = 1 - e^{-\lambda c}$$

therefore,

$$E[X|X < c] = \frac{\frac{1}{\lambda}\left(1 - e^{-\lambda c}\right) - c \cdot e^{-\lambda c}}{1 - e^{-\lambda c}} = \frac{1}{\lambda} - \frac{c\,e^{-\lambda c}}{1 - e^{-\lambda c}}$$

(b) Since X is exponential, it must be memoryless, therefore,

$$E[X|X > c] = \frac{1}{\lambda} + c$$

this can also be proved by integration

$$\frac{\int_c^\infty x\,f_X(x)\,dx}{\int_c^\infty f_X(x)\,dx} = \frac{(\frac{1}{\lambda} + c)e^{-\lambda c}}{e^{-\lambda c}} = \frac{1}{\lambda} + c$$

and we know

$$E[X] = \frac{1}{\lambda} \qquad P(X < c) = 1 - e^{-\lambda c} \qquad P(X > c) = e^{-\lambda c}$$

Substitute $E[X], P(X < c), P(X > c), E[X|X > c]$ into the identity

$$E[X] = E[X|X < c]P[X < c] + E[X|X > c]P[X > c]$$

we can solve that

$$E[X|X < c] = \frac{1}{\lambda} - \frac{c\,e^{-\lambda c}}{1 - e^{-\lambda c}}$$

2. **Solution for Q2:**

(a) the probability is 0.

(b) In this case, the only possibility for A still being post office after B and C leave is: service time for A,B,C are respectively, 3,1,1. Therefore, the desired probability is

$$p = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{27}$$

(c) The service time pdf is

$$f_T(t) = \mu\, e^{-\mu t}$$

Let T_A,T_B,T_C denote the service time for A,B and C respectively. Then

$$\begin{aligned}
p &= P(A \text{ leaves last}) = P(T_A > T_B + T_C) \\
&= E[P(T_A > t_b + t_c | T_B = t_b, T_C = t_c)] \\
&= \int_0^\infty \mu\, e^{-\mu t_b} \cdot \int_0^\infty \mu\, e^{-\mu t_c} \cdot \int_{t_b+t_c}^\infty \mu\, e^{-\mu t_a}\, dt_a\, dt_c\, dt_b \\
&= \int_0^\infty \mu\, e^{-\mu t_b} \cdot \int_0^\infty \mu\, e^{-\mu t_c} \cdot e^{-\mu(t_b+t_c)}\, dt_c\, dt_b = \frac{1}{4}
\end{aligned}$$

3. **Solution for Q3:**

The desired probability can be written as:

$$\begin{aligned}
p &= P(\text{machine 1 first fail}|\text{machine 1 works by time } t) \cdot P(\text{machine 1 works by time } t) \\
&\quad + P(\text{machine 1 first fail}|\text{machine 1 fails by time } t) \cdot P(\text{machine 1 fails by time } t) \\
&= \frac{\lambda_1}{\lambda_1 + \lambda_2} \cdot \int_t^\infty \lambda_1 e^{-\lambda_1 t}\, dt + 1 \cdot \int_0^t \lambda_1 e^{-\lambda_1 t}\, dt \\
&= e^{-\lambda_1 t} \cdot \frac{\lambda_1}{\lambda_1 + \lambda_2} + 1 - e^{-\lambda_1 t}
\end{aligned}$$

4. **Solution for Q4:**

Let

- T_A denote at time when A dies without a kidney (rate μ_A);
- T_B denote at time when B dies without a kidney (rate μ_B);
- T_1 denote the time when the first kidney arrive (rate λ);
- T_2 denote the time duration between the arrival of the first kidney and the second kidney.

(a)

$$\begin{aligned} p &= P(A \text{ obtains a new kidney}) = P(T_A > T_1) = E\left[P(T_A > c | T_1 = c)\right] \\ &= \int_0^\infty \lambda e^{-\lambda c} \cdot e^{-\mu_A c}\, dc = \int_0^\infty \lambda e^{-(\lambda+\mu_A)c}\, dc = \frac{\lambda}{\lambda+\mu_A} \end{aligned}$$

(b)

$$\begin{aligned} P &= P(B \text{ obtains a new kidney}) = P(B \text{ obtained the first kidney}) \\ &\quad + P(B \text{ obtained the second kidney}) = P(\text{event I}) + P(\text{event II}) \end{aligned}$$

where

$$\begin{aligned} P &= P(\text{event I}) = P(T_A < T_1 < T_B) = E[P(a < T_1 < b | T_A = a, T_B = b)] \\ &= \int_0^\infty \mu_A\, e^{-\mu_A a} \int_a^\infty \mu_B\, e^{-\mu_B b} \int_a^b \lambda\, e^{-\lambda t}\, dt\, db\, da \\ &= \int_0^\infty \mu_A\, e^{-\mu_A a} \int_a^\infty \mu_B\, e^{-\mu_B b}\, (e^{-\lambda a} - e^{-\lambda b})\, db\, da \\ &= \int_0^\infty \mu_A\, e^{-\mu_A a} \left(e^{-\lambda a}\, e^{-\mu_B b} - \frac{\mu_B}{\mu_B+\lambda} e^{-(\lambda+\mu_B)a} \right) da \\ &= \int_0^\infty \left(\mu_A - \frac{\mu_A\, \mu_B}{\mu_B+\lambda} \right) e^{-(\mu_A+\mu_B+\lambda)a}\, da \\ &= \frac{\mu_A \cdot \lambda}{\mu_B+\lambda} \cdot \frac{1}{\mu_A+\mu_B+\lambda} \end{aligned}$$

and

$$P = P(\text{event II}) = P(T_1 < T_A, T_1 < T_B) \cdot P(T_2 < T_B)$$

where $P(T_2 < T_B)$ can be interpreted that the event restart at point T_1, therefore,

$$P(T_2 < T_B) = \frac{\lambda}{\mu_B+\lambda}$$

while

$$\begin{aligned} P &= P(T_1 < T_A, T_1 < T_B) = E[P(T_A > a, T_B > a | T_1 = a)] \\ &= \int_0^\infty \lambda e^{-\lambda a} \int_a^\infty \mu_A\, e^{-\mu_A t} \int_a^\infty \mu_B\, e^{-\mu_B s}\, ds\, dt\, da \\ &= \int_0^\infty \lambda e^{-\lambda a}\, e^{-\mu_A a}\, e^{-\mu_B a}\, da = \frac{\lambda}{\lambda+\mu_A+\mu_B} \end{aligned}$$

Therefore,

$$P(B \text{ obtains a new kidney}) = \frac{\lambda^2 + \mu_A \lambda}{(\mu_B+\lambda)(\lambda+\mu_A+\mu_B)}$$

5. **Solution for Q5:**

Let T be uniformly distributed on $(0,1)$ (train interarrival time) and $X(t)$ be poisson arrival.

(a) since

$$E[X] = E[E[X|T]]$$

$$E[X|t] = E[X|T=t] = \sum_{n=0}^{\infty} n \cdot e^{-\lambda t} \cdot \frac{(\lambda\, t)^n}{n!} = \lambda\, t$$

therefore,

$$E[X] = E[\lambda T] = \int_0^1 \lambda\, t\, dt = \frac{\lambda}{2}$$

since $\lambda = 7$, $E[X] = 3.5$.

(b) since $Var[X] = E[X^2] - E^2[X]$

$$E[X^2|t] = \lambda\, t + \lambda^2\, t^2$$

therefore,

$$E[X^2] = E[\lambda\, T + \lambda^2\, T^2] = \int_0^1 (\lambda\, t + \lambda^2\, t^2)\, dt = \frac{\lambda}{2} + \frac{\lambda^2}{3}$$

we have

$$Var[X] = \frac{\lambda}{2} + \frac{\lambda^2}{3} - \left(\frac{\lambda}{2}\right)^2 = \frac{\lambda}{2} + \frac{\lambda^2}{12} = \frac{91}{12}$$

6. **Solution for Q6:**

Let T_k denote the elapsed time between the $(k-1)$st and the kth event. Then, T_k, $k = 1, 2, \cdots, n$ are independent identically distributed exponential random variables having mean $1/\lambda$.

(a) The time at which the last rider departs the car is

$$S_n = T_1 + T_2 + \cdots + T_n = \sum_{i=1}^{n} T_i$$

has a gamma distribution with parameter n and λ, which is given as

$$f_{S_n}(t) = \lambda\, e^{-\lambda t} \cdot \frac{(\lambda t)^{n-1}}{(n-1)!} \qquad t \geq 0$$

(b) Suppose a rider gets off at time s, then the probability that this rider has not arrived his home by time t is

$$\int_{t-s}^{\infty} \mu\, e^{-\mu\, x}\, dx = e^{-\mu(t-s)}$$

Given $S_n = t$, the time vector at which the $n-1$ riders get off are independent uniformly distributed over $[0, t]$. Hence, any one rider has not arrived his home by time t with probability

$$P = \int_0^t e^{-\mu(t-s)} \cdot \frac{1}{t}\, ds = \frac{1 - e^{-\mu t}}{\mu t}$$

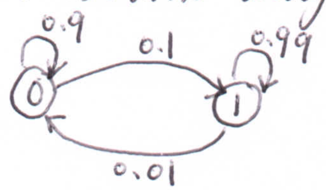
We consider $(n-1)$ riders, the probability that all arrived home is

$$P(\text{all arrived home}) = \left(1 - \frac{1 - e^{-\mu t}}{\mu t}\right)^{n-1}$$

Problem 12.3.3 Solution:

Let the state be 1 if the packet is error-free,
be 0 if the packet has an error.

The transition diagram is:



The transition matrix is:

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.01 & 0.99 \end{bmatrix}$$

Let π_0 — the steady probability at state 0
π_1 — the steady probability at state 1

From $\pi P = \pi$ and $\pi_0 + \pi_1 = 1$

$$\begin{cases} 0.9\pi_0 + 0.01\pi_1 = \pi_0 \\ \pi_0 + \pi_1 = 1 \end{cases}$$

We get $\pi_0 = \frac{1}{11}$, $\pi_1 = \frac{10}{11}$.

Therefore, the steady probability that a packet has an error is: $\frac{1}{11}$

Problem 12.5.6 Solution

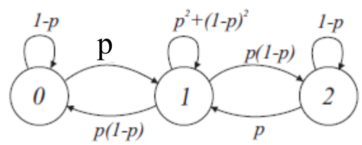
This system has three states:

0 front teller busy, rear teller idle

1 front teller busy, rear teller busy

2 front teller idle, rear teller busy

We will assume the units of time are seconds. Thus, if a teller is busy one second, the teller will become idle in th next second with probability $p = 1/120$. The Markov chain for this system is



We can solve this chain very easily for the stationary probability vector $\boldsymbol{\pi}$. In particular,

$$\pi_0 = (1-p)\pi_0 + p(1-p)\pi_1 \tag{1}$$

This implies that $\pi_0 = (1-p)\pi_1$. Similarly,

$$\pi_2 = (1-p)\pi_2 + p(1-p)\pi_1 \tag{2}$$

yields $\pi_2 = (1-p)\pi_1$. Hence, by applying $\pi_0 + \pi_1 + \pi_2 = 1$, we obtain

$$\pi_0 = \pi_2 = \frac{1-p}{3-2p} = 119/358 \tag{3}$$

$$\pi_1 = \frac{1}{3-2p} = 120/358 \tag{4}$$

The stationary probability that both tellers are busy is $\pi_1 = 120/358$.

Problem 10.3.4 Solution:

For $0 \leq x < 1$,

$$P[X(t) > x] = P\left[e^{-(t-T)}u(t-T) > x\right] \tag{1}$$
$$= P[t + \ln x < T \leq t] \tag{2}$$
$$= F_T(t) - F_T(t + \ln x) \tag{3}$$

Note that condition $T \leq t$ is needed to make sure that the pulse doesn't arrive after time t. The other condition $T > t + \ln x$ ensures that the pulse didn't arrrive too early and already decay too much. We can express these facts in terms of the CDF of $X(t)$.

$$F_{X(t)}(x) = 1 - P[X(t) > x] = \begin{cases} 0 & x < 0 \\ 1 + F_T(t + \ln x) - F_T(t) & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases} \tag{4}$$

We can take the derivative of the CDF to find the PDF. However, we need to keep in mind that the CDF has a jump discontinuity at $x = 0$. In particular, since $\ln 0 = -\infty$,

$$F_{X(t)}(0) = 1 + F_T(-\infty) - F_T(t) = 1 - F_T(t) \tag{5}$$

Hence, when we take a derivative, we will see an impulse at $x = 0$. The PDF of $X(t)$ is

$$f_{X(t)}(x) = \begin{cases} (1 - F_T(t))\delta(x) + f_T(t + \ln x)/x & 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$