Assignment 4 Solution – Part A

1. **Solution for Q1:**

   For $0 \le \mu < 1$, we have

$$\begin{aligned} P(U \le \mu) &= \int_{-\infty}^{\infty} P(U \le \mu | T = t) \cdot f_T(t)\, dt = \int_0^1 P(U \le \mu | T = t) \cdot 1\, dt \\ &= \int_0^{\mu} 1 \cdot 1\, dt + \int_{\mu}^{1} \frac{\mu}{t} \cdot 1\, dt = \mu + (\mu \ln t)\big|_{t=\mu}^{t=1} = \mu(1 - \ln \mu) \end{aligned} \tag{1}$$

   therefore, the complete CDF of U is

$$F_U(\mu) = \begin{cases} 0 & \mu < 0 \\ \mu(1 - \ln \mu) & 0 \le \mu < 1 \\ 1 & \mu \ge 1 \end{cases}$$

2. **Solution for Q2:**

   (a) For $x < 0$, $f_X(x) = 0$. For $x \ge 0$, we have

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy = \int_x^{\infty} \lambda^2 e^{-\lambda y}\, dy = \lambda e^{-\lambda x}$$

   We see that X is an exponential RV with $E[X] = 1/\lambda$. Given $X = x$, the conditional pdf of Y is

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x, y)}{f_X(x)} = \begin{cases} \lambda e^{-\lambda(y-x)} & y > x \\ 0 & \text{o.w.} \end{cases}$$

   To interpret this result, let $U = Y - X$ denote the interarrival time, the time between the arrival of the first and second calls. Given $X = x$, then U has the same pdf as X.

   (b) for $y \ge 0$,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx = \int_0^{y} \lambda^2 e^{-\lambda y}\, dx = \lambda^2\, y\, e^{-\lambda y}$$

   and

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)} = \begin{cases} \frac{1}{y} & 0 \le x < y \\ 0 & \text{o.w.} \end{cases}$$

   which implies that given $Y = y$, X is uniformly distributed in $[0, y]$.

3. **Solution for Q3:**

   (a) The joint characteristic function of X and Y is

$$\begin{aligned} \Phi_{X,Y}(\omega_1, \omega_2) &= E[e^{j\omega_1 X + j\omega_2 Y}] = E[e^{j\omega_1 X}] \cdot E[e^{j\omega_2 Y}] = \Phi_X(\omega_1) \cdot \Phi_Y(\omega_2) \\ &= e^{\lambda(e^{j\omega_1} - 1)} \cdot e^{j\mu\,\omega_2 - \sigma^2\,\omega_2^2/2} \end{aligned}$$

   (b) The characteristic function of Z is

$$\begin{aligned} \Phi_Z(\omega) &= E[e^{j\omega Z}] = E[e^{j\omega(X+Y)}] = \Phi_{X,Y}(\omega, \omega) \\ &= e^{\lambda(e^{j\omega} - 1) + j\mu\,\omega - \sigma^2\,\omega^2/2} \end{aligned}$$

Note: since $Z = X + Y$, and X and Y are independent, we have

$$\Phi_Z(\omega) = \Phi_X(\omega) \cdot \Phi_Y(\omega)$$

part (a) dealing with two-dimensional CF, while part (b) dealing with one-dimensional CF.

4. **Solution for Q4:**

(a) From the given joint mass function, we can have marginal pmf as

$$\begin{aligned}
P(Y=1) &= \frac{1}{9} + \frac{1}{3} + \frac{1}{9} = \frac{5}{9} \\
P(Y=2) &= \frac{1}{9} + 0 + \frac{1}{18} = \frac{1}{6} \\
P(Y=3) &= 0 + \frac{1}{6} + \frac{1}{9} = \frac{5}{18}
\end{aligned}$$

therefore, the conditional pmfs are:

$$\begin{aligned}
P_{X|Y}(X=1|Y=1) &= \frac{P(1,1)}{P_Y(y=1)} = \frac{1/9}{5/9} = \frac{1}{5} \\
P_{X|Y}(X=2|Y=1) &= \frac{P(2,1)}{P_Y(y=1)} = \frac{1/3}{5/9} = \frac{3}{5} \\
P_{X|Y}(X=3|Y=1) &= \frac{P(3,1)}{P_Y(y=1)} = \frac{1/9}{5/9} = \frac{1}{5}
\end{aligned}$$

and

$$E[X|Y=1] = 1 \times \frac{1}{5} + 2 \times \frac{3}{5} + 3 \times \frac{1}{5} = 2$$

Similarly,

$$\begin{aligned}
P_{X|Y}(X=1|Y=2) &= \frac{P(1,2)}{P_Y(y=2)} = \frac{1/9}{1/6} = \frac{2}{3} \\
P_{X|Y}(X=2|Y=2) &= 0 \qquad P_{X|Y}(X=3|Y=2) = \frac{P(3,2)}{P_Y(y=2)} = \frac{1/18}{1/6} = \frac{1}{3}
\end{aligned}$$

and

$$E[X|Y=2] = 1 \times \frac{2}{3} + 2 \times 0 + 3 \times \frac{1}{3} = \frac{5}{3}$$

For $Y = 3$,

$$\begin{aligned}
P_{X|Y}(X=1|Y=3) &= \frac{P(1,3)}{P_Y(y=3)} = 0 \qquad P_{X|Y}(X=2|Y=3) = \frac{1/6}{5/18} = \frac{3}{5} \\
P_{X|Y}(X=3|Y=3) &= \frac{2}{5}
\end{aligned}$$

and

$$E[X|Y=3] = 1 \times 0 + 2 \times \frac{3}{5} + 3 \times \frac{2}{5} = \frac{12}{5}$$

(b) From the fact that $E[X|Y=1] \neq E[X|Y=2] \neq E[X|Y=3]$, or from the fact that $P_{X|Y}(X=1|Y=1) \neq P_{X|Y}(X=1|Y=2) \neq P_X(X=1)$, we conclude that X and Y are not independent.

5. **Solution for Q5:**

The marginal density function of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx = \int_0^y \frac{1}{y}\, e^{-y}\, dx = e^{-y}$$

therefore, the conditional density is

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)} = \frac{\frac{1}{y}\, e^{-y}}{e^{-y}} = \frac{1}{y} \qquad (0 < x < y)$$

and

$$E[X^2|Y = y] = \frac{1}{y} \int_0^y x^2\, dx = \frac{y^2}{3}$$

6. **Solution for Q6:**

(a) According to the definition of RVs, X, N and T_i, we can have

$$X = \sum_{i=1}^{N} T_i$$

(b) Clearly, N is a RV with Geometric distribution with a pmf given as

$$P[N = k] = \left(\frac{2}{3}\right)^{k-1} \cdot \left(\frac{1}{3}\right) = p \cdot q^{n-1}$$

hence, $E[N] = \frac{1}{p} = 3$.

(c) Since T_N is the travel time correspondingly to the choice leading to freedom, it follows that $T_N = 2$ and so

$$E[T_N] = 2$$

(d) Given that $N = n$, the travel times $T_i, i = 1, 2, \cdots, n-1$ are each equally likely to be either 3 or 5 (since we know that a door leading to the mine is selected), whereas T_n is equal to 2 (since that choice leads to safety). Hence,

$$\begin{aligned} E\left[\sum_{i=1}^{N} T_i | N = n\right] &= E\left[\sum_{i=1}^{n-1} T_i | N = n\right] + E\left[T_n | N = n\right] \\ &= (3+5) \cdot \frac{1}{2} \cdot (n-1) + 2 = 4n - 2 \end{aligned}$$

(e)

$$\begin{aligned} E[X] &= E\left[\sum_{i=1}^{N} T_i\right] = E\left[E\left[\sum_{i=1}^{N} T_i | N\right]\right] = E[4N - 2] \\ &= 4 \cdot E[N] - 2 = 10 \end{aligned}$$

7. **Solution for Q7:**

(a) the state transition diagram is ignored. There are three classes: (i)state 0 is recurrent state; (ii) state 1 and 2 are recurrent states; (iii) state 3 and 4 are transient states.

(b) From $\pi P = \pi$, we obtain the equation set:

$$\begin{aligned} \pi_0 + \tfrac{1}{4}\pi_3 &= \pi_0 \\ \tfrac{3}{4}\pi_1 + \tfrac{1}{2}\pi_2 &= \pi_2 \\ \tfrac{1}{4}\pi_3 + \tfrac{1}{2}\pi_4 &= \pi_3 \\ \tfrac{1}{4}\pi_3 + \tfrac{1}{2}\pi_4 &= \pi_4 \end{aligned}$$

Solving the equation set, we obtain:

$$\pi_3 = 0 \quad \pi_2 = \frac{3}{2}\pi_1 \quad \pi_4 = 0$$

when the state transits to state 0, stationary distribution $\pi_1 = \pi_2 = 0$; when the state transits to state 1 and 2, then from $\pi_2 = \frac{3}{2}\pi_1$ and $\pi_1 + \pi_2 = 1$, we obtain $\pi_1 = \frac{2}{5}$ and $\pi_2 = \frac{3}{5}$.

(c) the transition probability matrix for transient state is

$$P_T = \begin{bmatrix} 1/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}$$

therefore,

$$S = (I - P_T)^{-1} = \begin{bmatrix} 3/4 & -1/4 \\ -1/2 & 1/2 \end{bmatrix}^{-1} = \begin{bmatrix} 2.0 & 1.0 \\ 2.0 & 3.0 \end{bmatrix}$$

(d)

$$\begin{aligned} P &= P(X_5 = 2|X_3 = 1) = \sum_k P(X_5 = 2|X_4 = k) \cdot P(X_4 = k|X_3 = 1) \\ &= P(X_5 = 2|X_4 = 1) \cdot P(X_4 = 1|X_3 = 1) + P(X_5 = 2|X_4 = 2) \cdot P(X_4 = 2|X_3 = 1) \\ &= P_{12} \cdot P_{11} + P_{22} \cdot P_{12} = \frac{3}{4} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{3}{4} = \frac{9}{16} \end{aligned}$$

8. **Solution for Q8:**

Let the state be 0 of the last two trials were both successes (SS) and be 1 if last trial a success and the one before is a failure (FS), be 2 if the last trial was a failure. The transition matrix of this Markov chain is

$$P = \begin{bmatrix} 0.8 & 0 & 0.2 \\ 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

Then we have equation set

$$\begin{aligned} 0.8\pi_0 + 0.5\pi_1 &= \pi_0 \\ 0.5\pi_2 &= \pi_1 \\ \pi_0 + \pi_1 + \pi_2 &= 1 \end{aligned}$$

we can solve that

$$\pi_0 + 0.4\pi_0 + 0.8\pi_0 = 1$$

which implies

$$\pi_0 = \frac{5}{11} \quad \pi_1 = \frac{2}{11} \quad \pi_2 = \frac{4}{11}$$

Consequently, the proportion of trials that are successes is

$$0.8\pi_0 + 0.5(1 - \pi_0) = \frac{7}{11}$$

10. **Solution for Q10:**

The transition probability matrix is

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 & 0 \end{bmatrix}$$

The equation set is $(\pi P = \pi)$

$$\begin{aligned} 1/2\pi_2 + 1/3\pi_3 + 1/4\pi_4 &= \pi_1 \\ 1/3\pi_3 + 1/4\pi_4 &= \pi_2 \\ 1/4\pi_4 &= \pi_3 \\ \pi_0 &= \pi_4 \\ \pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 &= 1 \end{aligned}$$

solving the equation, we obtain

$$\pi = \left[\frac{12}{37}, \frac{6}{37}, \frac{4}{37}, \frac{3}{37}, \frac{12}{37}\right]$$

9. **Solution for Q9:**

S F ← last trial

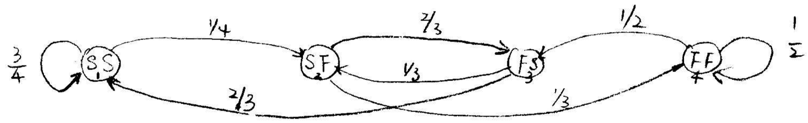
└ next to last

If new trial is success, state changes to $\Longrightarrow$ F S

Q.9 (Ross 4.28)

Let S denotes a success trial and F a failure.

Then SS, SF, FS, FF form a four state Markov chain, with the transition diagram:



The transition martrix is,

$$P = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

from $P'\pi' = \pi'$ & $\Sigma \pi_j = 1$.

$$\begin{bmatrix} \frac{3}{4} & 0 & \frac{2}{3} & 0 \\ \frac{1}{4} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{2} \\ 0 & \frac{1}{3} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{bmatrix} = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{bmatrix}$$

Solve this equation, we have:

$$\pi = \left[\frac{1}{2}, \frac{3}{16}, \frac{3}{16}, \frac{1}{8} \right]$$

$$\therefore \lim_{n\to\infty} P(\text{success on the nth trial}) = \pi_1 \cdot 1 + \pi_2 \cdot \frac{1}{2} + \pi_3 \cdot \frac{1}{2}$$

$$= \frac{1}{2} + \frac{3}{16} = \frac{11}{16}$$

P8/12

Solutions for the questions from the textbook:

Problem 4.1.6 Solution

The given function is

$$F_{X,Y}(x,y) = \begin{cases} 1 - e^{-(x+y)} & x, y \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

First, we find the CDF $F_X(x)$ and $F_Y(y)$.

$$F_X(x) = F_{X,Y}(x,\infty) = \begin{cases} 1 & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

$$F_Y(y) = F_{X,Y}(\infty,y) = \begin{cases} 1 & y \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Hence, for any $x \geq 0$ or $y \geq 0$,

$$P[X > x] = 0 \qquad P[Y > y] = 0 \tag{4}$$

For $x \geq 0$ and $y \geq 0$, this implies

$$P[\{X > x\} \cup \{Y > y\}] \leq P[X > x] + P[Y > y] = 0 \tag{5}$$

However,

$$P[\{X > x\} \cup \{Y > y\}] = 1 - P[X \leq x, Y \leq y] = 1 - (1 - e^{-(x+y)}) = e^{-(x+y)} \tag{6}$$

Thus, we have the contradiction that $e^{-(x+y)} \leq 0$ for all $x, y \geq 0$. We can conclude that the given function is not a valid CDF.

Problem 4.3.5 Solution

For $n = 0, 1, \ldots$, the marginal PMF of N is

$$P_N(n) = \sum_k P_{N,K}(n,k) = \sum_{k=0}^{n} \frac{100^n e^{-100}}{(n+1)!} = \frac{100^n e^{-100}}{n!} \tag{1}$$

For $k = 0, 1, \ldots$, the marginal PMF of K is

$$\begin{aligned}
P_K(k) &= \sum_{n=k}^{\infty} \frac{100^n e^{-100}}{(n+1)!} && (2)\\
&= \frac{1}{100} \sum_{n=k}^{\infty} \frac{100^{n+1} e^{-100}}{(n+1)!} && (3)\\
&= \frac{1}{100} \sum_{n=k}^{\infty} P_N(n+1) && (4)\\
&= P[N > k]/100 && (5)
\end{aligned}$$

Problem 4.4.2 Solution

Given the joint PDF

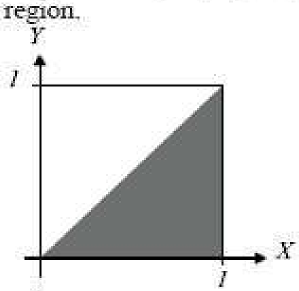
$$f_{X,Y}(x,y) = \begin{cases} cxy^2 & 0 \le x, y \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) To find the constant c integrate $f_{X,Y}(x,y)$ over the all possible values of X and Y to get

$$1 = \int_0^1 \int_0^1 cxy^2 \, dx \, dy = c/6 \tag{2}$$

Therefore $c = 6$.

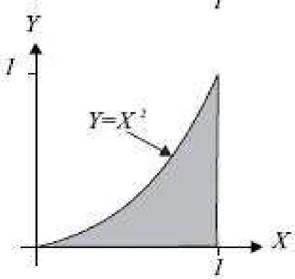
(b) The probability $P[X \ge Y]$ is the integral of the joint PDF $f_{X,Y}(x,y)$ over the indicated shaded region.



$$P[X \ge Y] = \int_0^1 \int_0^x 6xy^2 \, dy \, dx \tag{3}$$

$$= \int_0^1 2x^4 \, dx \tag{4}$$

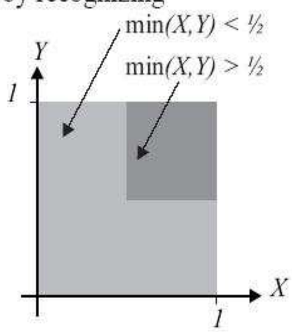
$$= 2/5 \tag{5}$$



Similarly, to find $P[Y \le X^2]$ we can integrate over the region shown in the figure.

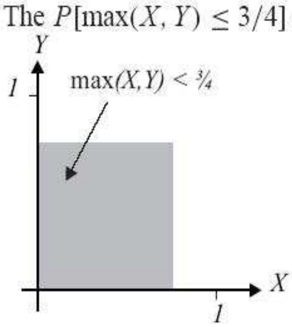
$$P[Y \le X^2] = \int_0^1 \int_0^{x^2} 6xy^2 \, dy \, dx = 1/4 \tag{6}$$

(c) Here we can choose to either integrate $f_{X,Y}(x,y)$ over the lighter shaded region, which would require the evaluation of two integrals, or we can perform one integral over the darker region by recognizing



$$P[\min(X,Y) \le 1/2] = 1 - P[\min(X,Y) > 1/2] \tag{7}$$
$$= 1 - \int_{1/2}^{1}\int_{1/2}^{1} 6xy^2\,dx\,dy \tag{8}$$
$$= 1 - \int_{1/2}^{1} \frac{9y^2}{4}\,dy = \frac{11}{32} \tag{9}$$

(d) The $P[\max(X,Y) \le 3/4]$ can be found be integrating over the shaded region shown below.

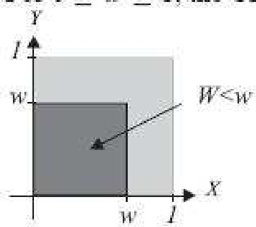


$$P[\max(X,Y) \le 3/4] = P[X \le 3/4, Y \le 3/4] \tag{10}$$
$$= \int_0^{\frac{3}{4}}\int_0^{\frac{3}{4}} 6xy^2\,dx\,dy \tag{11}$$
$$= (x^2\big|_0^{3/4})(y^3\big|_0^{3/4}) \tag{12}$$
$$= (3/4)^5 = 0.237 \tag{13}$$

Problem 4.6.6 Solution

(a) The minimum value of W is $W = 0$, which occurs when $X = 0$ and $Y = 0$. The maximum value of W is $W = 1$, which occurs when $X = 1$ or $Y = 1$. The range of W is $S_W = \{w|0 \leq w \leq 1\}$.

(b) For $0 \leq w \leq 1$, the CDF of W is



$$F_W(w) = P[\max(X, Y) \leq w] \quad (1)$$
$$= P[X \leq w, Y \leq w] \quad (2)$$
$$= \int_0^w \int_0^w f_{X,Y}(x, y)\, dy\, dx \quad (3)$$

Substituting $f_{X,Y}(x, y) = x + y$ yields

$$F_W(w) = \int_0^w \int_0^w (x + y)\, dy\, dx = \int_0^w \left(xy + \frac{y^2}{2}\Big|_{y=0}^{y=w}\right) dx = \int_0^w (wx + w^2/2)\, dx = w^3 \quad (4)$$

The complete expression for the CDF is

$$F_W(w) = \begin{cases} 0 & w < 0 \\ w^3 & 0 \leq w \leq 1 \\ 1 & \text{otherwise} \end{cases} \quad (5)$$

The PDF of W is found by differentiating the CDF.

$$f_Y(y) = \frac{dF_W(w)}{dw} = \begin{cases} 3w^2 & 0 \leq w \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad (6)$$

Problem 4.8.6 Solution

Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} (4x+2y)/3 & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) The probability that $Y \le 1/2$ is

$$P[A] = P[Y \le 1/2] = \iint_{y \le 1/2} f_{X,Y}(x,y)\,dy\,dx \tag{2}$$

$$= \int_0^1 \int_0^{1/2} \frac{4x+2y}{3}\,dy\,dx \tag{3}$$

$$= \int_0^1 \left.\frac{4xy+y^2}{3}\right|_{y=0}^{y=1/2} dx \tag{4}$$

$$= \int_0^1 \frac{2x+1/4}{3}\,dx = \left.\frac{x^2}{3} + \frac{x}{12}\right|_0^1 = \frac{5}{12} \tag{5}$$

(b) The conditional joint PDF of X and Y given A is

$$f_{X,Y|A}(x,y) = \begin{cases} \frac{f_{X,Y}(x,y)}{P[A]} & (x,y) \in A \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 8(2x+y)/5 & 0 \le x \le 1, 0 \le y \le 1/2 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

For $0 \le x \le 1$, the PDF of X given A is

$$f_{X|A}(x) = \int_{-\infty}^{\infty} f_{X,Y|A}(x,y)\,dy = \frac{8}{5}\int_0^{1/2} (2x+y)\,dy = \frac{8}{5}(2xy + \frac{y^2}{2})\Big|_{y=0}^{y=1/2} = \frac{8x+1}{5} \tag{7}$$

The complete expression is

$$f_{X|A}(x) = \begin{cases} (8x+1)/5 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

For $0 \le y \le 1/2$, the conditional marginal PDF of Y given A is

$$f_{Y|A}(y) = \int_{-\infty}^{\infty} f_{X,Y|A}(x,y)\,dx = \frac{8}{5}\int_0^1 (2x+y)\,dx \tag{9}$$

$$= \left.\frac{8x^2+8xy}{5}\right|_{x=0}^{x=1} \tag{10}$$

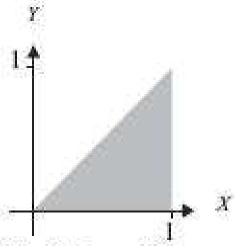
$$= \frac{8y+8}{5} \tag{11}$$

The complete expression is

$$f_{Y|A}(y) = \begin{cases} (8y+8)/5 & 0 \le y \le 1/2 \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

Problem 4.9.4 Solution

Random variables X and Y have joint PDF



$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

For $0 \le y \le 1$,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx = \int_y^1 2\, dx = 2(1-y) \quad (2)$$

Also, for $y < 0$ or $y > 1$, $f_Y(y) = 0$. The complete expression for the marginal PDF is

$$f_Y(y) = \begin{cases} 2(1-y) & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

By Theorem 4.24, the conditional PDF of X given Y is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \frac{1}{1-y} & y \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

That is, since $Y \le X \le 1$, X is uniform over $[y, 1]$ when $Y = y$. The conditional expectation of X given $Y = y$ can be calculated as

$$E[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\, dx = \int_y^1 \frac{x}{1-y}\, dx = \left.\frac{x^2}{2(1-y)}\right|_y^1 = \frac{1+y}{2} \quad (5)$$

In fact, since we know that the conditional PDF of X is uniform over $[y, 1]$ when $Y = y$, it wasn't really necessary to perform the calculation.

Problem 4.9.9 Solution

Random variables N and K have the joint PMF

$$P_{N,K}(n,k) = \begin{cases} \frac{100^n e^{-100}}{(n+1)!} & k = 0,1,\ldots,n; \\ & n = 0,1,\ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

We can find the marginal PMF for N by summing over all possible K. For $n \geq 0$,

$$P_N(n) = \sum_{k=0}^{n} \frac{100^n e^{-100}}{(n+1)!} = \frac{100^n e^{-100}}{n!} \tag{2}$$

We see that N has a Poisson PMF with expected value 100. For $n \geq 0$, the conditional PMF of K given $N = n$ is

$$P_{K|N}(k|n) = \frac{P_{N,K}(n,k)}{P_N(n)} = \begin{cases} 1/(n+1) & k = 0,1,\ldots,n \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

That is, given $N = n$, K has a discrete uniform PMF over $\{0,1,\ldots,n\}$. Thus,

$$E[K|N=n] = \sum_{k=0}^{n} k/(n+1) = n/2 \tag{4}$$

We can conclude that $E[K|N] = N/2$. Thus, by Theorem 4.25,

$$E[K] = E[E[K|N]] = E[N/2] = 50. \tag{5}$$