Assignment 3 Solution

1. **Solution for Q1:**

   From $Y = a/X$, we have
   $$\frac{dy}{dx} = -\frac{a}{x^2} \qquad x = \frac{a}{y}$$
   and
   $$\left|\frac{dy}{dx}\right| = \left|\frac{y^2}{a}\right| \qquad f_Y(y) = \sum_i \frac{1}{|dy/dx|_i} f_X(x_i)$$
   therefore,
   $$\text{when } a > 0 \qquad f_Y(y) = \frac{a}{y^2}\, f_X\left(\frac{a}{y}\right) \tag{1}$$
   $$\text{when } a < 0 \qquad f_Y(y) = -\frac{a}{y^2}\, f_X\left(\frac{a}{y}\right) \tag{2}$$

2. **Solution for Q2:**

   Since $Y = cX^2$,
   $$\frac{dy}{dx} = 2cx \qquad \text{and two roots at } x_1 = \sqrt{\frac{y}{c}}, \quad x_2 = -\sqrt{\frac{y}{c}}$$
   Therefore,
   $$f_Y(y) = \sum_i \frac{1}{|dy/dx|_i} f_X(x_i) = \frac{1}{2c\sqrt{\frac{y}{c}}}\left(f_X(\sqrt{\frac{y}{c}}) + f_X(-\sqrt{\frac{y}{c}})\right)$$
   $$= \frac{f_X(\sqrt{\frac{y}{c}}) + f_X(-\sqrt{\frac{y}{c}})}{2\sqrt{cy}} \tag{3}$$
   while
   $$f_X(x) = \frac{1}{2a} \quad (-a < x < a)$$
   while implies $0 < y < ca^2$. Therefore,
   $$f_Y(y) = \frac{1/2a + 1/2a}{2\sqrt{cy}} = \frac{1}{2a\sqrt{cy}} \quad 0 < y < ca^2$$

3. **Solution for Q3:**
   $$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \quad \text{and } Y = g(X) = 5X^2$$
   $$E[Y] = \int_{-\infty}^{\infty} f(x) f_X(x)\, dx = \int_{-\infty}^{\infty} 5x^2 \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}\, dx = 5\int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}\, dx = 5\sigma^2 = 45$$
   The last integral is the variance of zero mean Gaussian RV, we can obtain the result of the integral directly without integration.

4. **Solution for Q4:**

(a)

$$P(X=-1) = P(\xi=-1) = \frac{1}{5} \qquad P(Y=1) = P(\xi=-1) + P(\xi=1) = \frac{2}{5}$$

while

$$P(X=-1, Y=1) = P(\xi=-1) = \frac{1}{5}$$

therefore,

$$P(X=-1)\,P(Y=1) = \frac{1}{5}\cdot\frac{2}{5} = \frac{2}{25} \neq P(X=-1, Y=1)$$

hence, X and Y are dependent RVs.

(b)

$$E[X] = \sum x_i P(x_i) = \sum_{i=1}^{5} \xi_i P(\xi_i) = \frac{1}{5}(-1-1/2+0+1/2+1) = 0$$

$$E[Y] = \sum y_i P(y_i) = \sum_{i=1}^{5} \xi_i^2 P(\xi_i) = \frac{1}{5}(1+1/4+0+1/4+1) = 1/2$$

$$E[XY] = \sum xyP(x,y) = \sum_{i=1}^{5} \xi^3 P(\xi) = \frac{1}{5}(-1-1/8+0+1/8+1) = 0$$

Since $E[XY] = E[X]\cdot E[Y]$, X and Y are uncorrelated RVs.

5. **Solution for Q5:**

Since,

$$X_n = Z_n - aZ_{n-1} \qquad |a| < 1$$

$$E[Z_n] = 0 \qquad E[Z_n Z_j] = 0 \qquad E[Z_n^2] = \sigma^2$$

We have,

$$\begin{aligned} R_n(k) &= E[X_n X_{n-k}] = E[(Z_n - aZ_{n-1})(Z_{n-k} - aZ_{n-k-1})] \\ &= E[Z_n Z_{n-k} - aZ_n Z_{n-k-1} - aZ_{n-1}Z_{n-k} + a^2 Z_{n-1} Z_{n-k-1}] \end{aligned} \tag{4}$$

$$R_n(0) = E[Z_n^2 - 2aZ_{n-1}Z_n + a^2 Z_{n-1}^2 = (1+a^2)\sigma^2$$

$$R_n(-1) = R_n(1) = E[Z_n Z_{n-1} - aZ_n Z_{n-2} - aZ_{n-1}^2 + a^2 Z_{n-1} Z_{n-2} = -a\sigma^2$$

When $k > 1$, the expectation of each term in (4) equals zero. Therefore,

$$R_n(k) = \begin{cases} (1+a^2)\sigma^2 & k = 0 \\ -a\sigma^2 & k = \pm 1 \\ 0 & o.w. \end{cases}$$

6. **Solution for Q6:**

   To find the constant c, we apply $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^2\int_0^1 cxydxdy = \frac{c}{2}\int_0^2 y\,dy = c$$

   therefore, $c = 1$. To calculate $P(A)$, we write

$$P[A] = \int\int_A f(x,y)\,dx\,dy$$

   we use polar coordinates, using $x = r\cos\theta$ and $y = r\sin\theta$, and $dx\,dy = rdr\,d\theta$,

$$P[A] = \int_0^{\pi/2}\int_0^1 r^2\sin\theta\,\cos\theta\,r\,dr\,d\theta = \int_0^1 r^3\,dr\,\int_0^{\pi/2}\sin\theta\,\cos\theta\,d\theta = 1/8$$

7. **Solution for Q7:**

   First, find the constant c,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^{2\pi}\int_0^1 c\cdot rdr\,d\theta = 1$$

   Therefore, $c = 1/\pi$. The probability that the distance from the origin is less than x is

$$\int_0^{2\pi}\int_0^x 1/\pi\cdot rdr\,d\theta = x^2$$

8. **Solution for Q8:**

   Proof:

$$\begin{aligned} P[X\le Y] &= \int\int_{x\le y} f_X(x)f_Y(y)\,dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{y} f_X(x)f_Y(y)\,dxdy \\ &= \int_{-\infty}^{\infty} f_Y(y)\int_{-\infty}^{y} f_X(x)\,dx\;dy = \int_{-\infty}^{\infty} f_Y(y)F_X(y)dy \end{aligned} \quad (5)$$

9. **Solution for Q9:**

   Proof:

$$E[XY] = E[X]E[Y] = \mu_X\mu_Y$$
$$E[X^2] = \mu_X^2 + \sigma_X^2 \quad \text{and} \quad E[Y^2] = \mu_Y^2 + \sigma_Y^2$$

   therefore,

$$E[(XY)^2] = E[X^2]\cdot E[Y^2] = (\mu_X^2 + \sigma_X^2)(\mu_Y^2 + \sigma_Y^2)$$

   hence,

$$Var(XY) = E[(XY)^2] - E^2[XY] = (\mu_X^2 + \sigma_X^2)(\mu_Y^2 + \sigma_Y^2) - \mu_X^2\mu_Y^2 = \sigma_X^2\sigma_Y^2 + \mu_Y^2\sigma_X^2 + \mu_X^2\sigma_Y^2.$$

10. **Solution for Q10:**

Let RV. $U = X + Y$ and $V = X - Y$, then the joint moment generating function is

$$\begin{aligned}\Phi(\omega_1, \omega_2) &= E\left(e^{\omega_1 u} + e^{\omega_2 v}\right) = E\left(e^{\omega_1 X + \omega_1 Y + \omega_2 X - \omega_2 Y}\right) \\ &= E\left(e^{(\omega_1+\omega_2)X + (\omega_1-\omega_2)Y}\right)\end{aligned}$$

since X and Y are independent Gaussian $\sim N(\mu, \sigma^2)$, therefore,

$$\begin{aligned}\Phi(\omega_1, \omega_2) &= E\left[e^{(\omega_1+\omega_2)\mathrm{X}}\right] \cdot E\left[e^{(\omega_1-\omega_2)Y}\right] \\ &= \exp\left[\mu(\omega_1+\omega_2) + \frac{\sigma^2(\omega_1+\omega_2)^2}{2}\right] \cdot \exp\left[\mu(\omega_1-\omega_2) + \frac{\sigma^2(\omega_1-\omega_2)^2}{2}\right] \\ &= \exp\left[2\mu\omega_1 + \frac{\sigma^2 \cdot 2\omega_1^2}{2} + \frac{\sigma^2 \cdot 2\omega_2^2}{2}\right] \\ &= \exp\left[2\mu\omega_1 + \frac{(2\sigma^2)\cdot\omega_1^2}{2}\right] \cdot \exp\left[\frac{(2\sigma^2)\cdot\omega_2^2}{2}\right] \\ &= \Phi(\omega_1)\cdot\Phi(\omega_2) \qquad (6)\end{aligned}$$

where we have applied the fact that $U \sim N(2\mu, 2\sigma^2)$ and $V \sim N(0, 2\sigma^2)$. From (6), we can obtain the result that U and V are independent.

11. **Solution for Q11:**

(a)

$$\Phi_{X_i}(\omega_i) = \Phi(0, 0, \cdots, 0, \omega_i, 0, \cdots, 0)$$

with ω_i in the ith place, and all others to be 0.

(b) If independent, then the joint CF is

$$E\left[e^{j\sum \omega_i X_i}\right] = E\left[e^{j\omega_1 X_1} \cdot e^{j\omega_2 X_2} \cdots\right] = \prod_i E\left[e^{j\omega_i X_i}\right] = \prod_i \Phi_{X_i}(\omega_i) \qquad (7)$$

On the other hand, if the above is satisfied, then the joint CF is that of the sum of n independent random variables, and the ith of which has the same distribution as X_i. As the joint CF uniquely determines the joint distribution, the results follows.

12. **Solution for Q12:**

It is easy to see that Y takes values only in the interval $(9/11, 9/9)$ =(0.8182,1), as shown in Fig.1. Therefore, the value of the distribution function, $F_Y(v)$, is zero for $v < 0.8182$ and is 1 for $v > 1$. What happens between these two values can be inferred from the figure, where a line of a value of $v = 0.918$ is shown, so that we see that for Y to fall below this value, the value of Y has to be higher than $9/v$, a point obtained by finding the inverse value of the function $9/X$. The resulting distribution function is obtained as follows:

$$F_Y(v) = P\left(\frac{9}{X} \leq v\right) = P\left(X \geq \frac{9}{v}\right) = (11 - 9/v)/2 = 5.5 - 4.5/v \qquad 0.8182 < v < 1$$

and

$$F_Y(v) = 0, \qquad v < 0.8182, \text{ or } v > 1$$

The CDF of Y is shown in top figure in FIg.2. The density function of Y is obtained by taking the derivative of the distribution function; for the values in the range (0.8182,1), it has the expression:

$$f_Y(v) = 4.5/v^2, \qquad 0.8182 < v < 1$$

and is zero elsewhere, as shown in the bottom figure in Fig.2. It is interesting to note that even though X is uniformly distributed in the interval (9,11), the resulting current is not uniformly distributed.

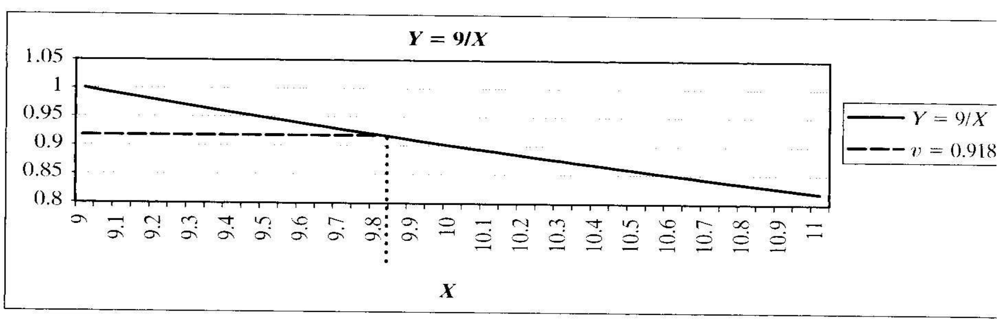

Figure 1: Y as a function of X, and a line showing one value of v (figure for Q12).

13. **Solution for Q13:**

    (a) when $v < -a$, we obtain no value of Y, since Y takes values only between $-a$ and $+a$, which implies

    $$F_Y(v) = P(Y \leq v) = 0 \qquad v < -a$$

    (b) Similarly, when $v > +a$, all values of Y will fall below the value of v, which means

    $$F_Y(v) = P(Y \leq v) = 1 \qquad v > +a$$

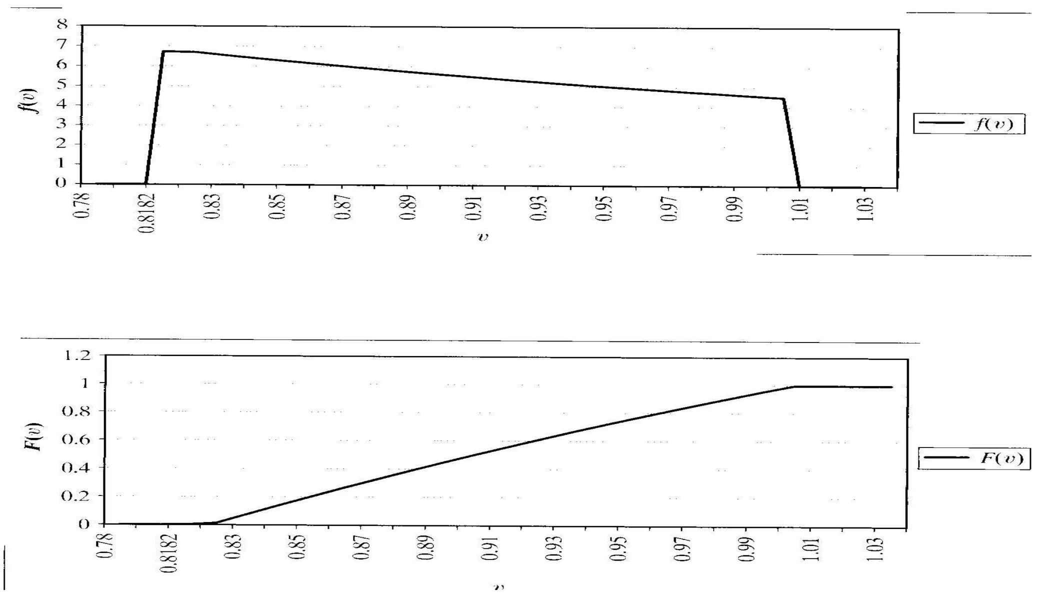

Figure 2: The probability density function and cumulative distribution function of Y (figure for Q12).

(c) When $-a < v < +a$, we have the following relation for the probability (note that a is positive here):

$$F_Y(v) = P(Y \leq v) = P(aX \leq v) = P(X \leq v/a) = F_X(v/a) \qquad -a < v < +a$$

Differentiate the CDF of Y, we obtain the value of the pdf of Y in the interval $(-a, +a)$ as

$$f_Y(v) = \mathrm{f}_X(v/a)/a = \frac{1}{a\,0.5\sqrt{2\pi}} \exp\left\{-\frac{v^2}{2(a\,0.5)^2}\right\} \qquad -a < v < +a$$

Note that the pdf is valid only for values of Y inside the interval $(-a, +a)$. Also, since we have a jump in $F_Y(v)$ at $v = -a$ and $v = +a$, Y has a mixed distribution. The value of the jumps may be obtained from

$$P(Y = -a) = F_Y(-a) - F_Y(-a^-) = F_X(-a/a) - 0 = F_X(-1) = \Phi(-2) = 0.0227$$

and

$$P(Y = a) = F_Y(a) - F_Y(a^-) = 1 - F_X(a/a) = 1 - F_X(1) = 1 - \Phi(2) = 1 - 0.9773 = 0.0227$$

14. **Solution for Q14:**

The pdf of Cauchy RV X with parameter α is give as

$$f_X(x) = \frac{\alpha/\pi}{(x^2 + \alpha^2)} \qquad -\infty < x < \infty$$

The corresponding CF of X is given as

$$\Phi_X(\omega) = e^{-\alpha|\omega|}$$

Similarly, the CF of Y with parameter β is given as

$$\Phi_Y(\omega) = e^{-\beta|\omega|}$$

Since X and Y are independent, the CF of the sum RV $Z = X + Y$ is

$$\Phi_Z(\omega) = \Phi_X(\omega) \cdot \Phi_Y(\omega) = e^{-(\alpha+\beta)|\omega|}$$

which is a CF of a Cauchy RV with parameter $\alpha + \beta$, therefore, the cdf of Z is given as

$$f_Z(z) = \frac{(\alpha + \beta)/\pi}{(z^2 + (\alpha + \beta)^2)} \qquad -\infty < z < \infty$$