Assignment 2 Solution

1. We place at random n points in the interval (0,1) and we denote by random variables X and Y the distance from the origin to the first and the last points respectively. Find $F_X(x), F_Y(y)$ and $F_{X,Y}(x,y)$.

   Solution: Let $X = \min Z_i$, and $Y = \max Z_i$, where Z_i is i.i.d uniformly distributed in the interval $(0,1)$. Then, the CDF of X is

$$\begin{aligned} F_X(x) &= P(X \leq x) = P(\min Z_i \leq x) = P\{\text{at least one point in the interval}(0,x)\} \\ &= 1 - P\{\text{all } n \text{ points are in the interval}(x,1)\} \\ &= 1 - (1-x)^n \end{aligned}$$

   The CDF of Y is

$$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(\max Z_i \leq y) \\ &= P\{\text{all } n \text{ points are in the interval}(0,y)\} \\ &= y^n \end{aligned}$$

   The joint CDF is

$$F_{X,Y}(x,y) = P\{\min Z_i \leq x, \max Z_i \leq y\}$$

   (a) If $x > y$,

$$F_{X,Y}(x,y) = P\{\max Z_i \leq y\} = y^n$$

   (b) If $x \leq y$,

$$\begin{aligned} F_{X,Y}(x,y) &= P\{\text{all } n \text{ points in } (0,y)\} - P\{\text{all points in } (x,y)\ \\ &= y^n - (y-x)^n \end{aligned}$$

   (c) The complete joint CDF is

$$F_{X,Y}(x,y) = \begin{cases} y^n - (y-x)^n & x \leq y \\ y^n & x > y \end{cases}$$

2. **Problem 2 Solution:**

$$C = A \cap B = \{1 \leq X \leq 2.5\}$$

   For $x \geq 0$,

$$F_X(x) = \int_0^x \frac{1}{2}\, e^{-\alpha/2}\, d\alpha = \left[1 - e^{-x/2}\right] \qquad x \geq 0$$

   (a)

$$P(A) = F_X(3) - F_X(1) = e^{-1/2} - e^{-3/2} = 0.3834$$

(b)

$$P(B) = F_X(2.5) = 1 - e^{-1.25} = 0.7135$$

(c)

$$P(C) = F_X(2.5) - F_X(1) = e^{-1/2} - e^{-1.25} = 0.3200$$

3. **Problem 3 Solution:**

   (a) The probability of A is

$$\begin{aligned} P(A) &= P\{1000 < X < 3300\} = P\left\{\frac{1000-4000}{1000} < Y < \frac{3300-4000}{1000}\right\} \\ &= P\{-3.0 < Y < -0.7\} \end{aligned}$$

   where Y is normalized Gaussian therefore,

$$\begin{aligned} P(A) &= \Phi(-0.7) - \Phi(-3.0) = [1-\Phi(0.7)] - [1-\Phi(3.0)] = \Phi(3) - \Phi(0.7) \\ &= 0.9987 - 0.7580 = 0.2407 \end{aligned}$$

   (b) The probability of B is

$$\begin{aligned} P(B) &= P\{2000 < X < 4200\} = P\left\{\frac{2000-4000}{1000} < Y < \frac{4200-4000}{1000}\right\} \\ &= P\{-2 < Y < 0.2\} = \Phi(0.2) - \Phi(-2) \\ &= \Phi(0.2) + \Phi(2) - 1 = 0.5793 + 0.9772 - 1 = 0.5565 \end{aligned}$$

   (c) The probability of both correct is

$$\begin{aligned} P(\text{both correct}) &= P(A \cap B) = P\{2000 < X < 3300\} \\ &= \Phi\left(\frac{3300-4000}{1000}\right) - \Phi\left(\frac{2000-4000}{1000}\right) = \Phi(-0.7) - \Phi(-2.0) \\ &= \Phi(2.0) - \Phi(0.7) = 0.9772 - 0.7580 = 0.2192 \end{aligned}$$

4. **Problem 4 Solution:** Poisson distribution is

$$P(X = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

   (a) Substitute $\lambda = 4$ and integer k values, we can obtain the pmf as below when $i = 0, 1, \cdots, 8$,

$$P(i) = [0.0183, 0.0733, 0.1465, 0.1954, 0.1954, 0.1563, 0.1042, 0.0595, 0.0298]$$

   the corresponding CDF are given as

$$F(i) = [0.0183, 0.0916, 0.2381, 0.4335, 0.6289, 0.7852, 0.8894, 0.9489, 0.9787]$$

   where jump points are $i = 0, 1, \cdots, 8$

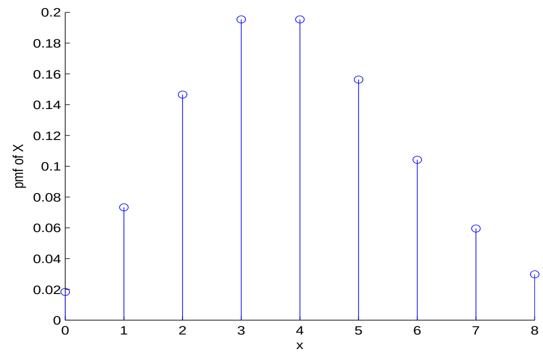

Figure 1: Probability mass function for Prob. 4

(b)

$$P(0 \le X \le 5) = \sum_{k=0}^{5} P(X = k) = F_X(5) = 0.7852$$

5. **Problem 5 Solution:**

(a) The mean of X is

$$E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx = \int_{-4}^{4} x \cdot \frac{\pi}{16} \cos\left(\frac{\pi x}{8}\right)\, dx = 0$$

where the integral is an odd function, and the result is zero when integrate over a symmetrical interval.

(b) The second moment is

$$\begin{aligned} E[X^2] &= \int_{-4}^{4} x^2 \cdot \frac{\pi}{16} \cos\left(\frac{\pi x}{8}\right)\, dx = \frac{1}{2}\int_{-4}^{4} x^2 d\left(\sin\frac{\pi x}{8}\right) \\ &= \frac{1}{2}\, x^2 \cdot \sin\left(\frac{\pi x}{8}\right)|_{-4}^{4} - \int_{-4}^{4} \sin\left(\frac{\pi x}{8}\right) x\, dx \\ &= 16 + \frac{8}{\pi}\int_{-4}^{4} x\, d\left(\cos\frac{\pi x}{8}\right) \\ &= 16 + \frac{8}{\pi}\, x \cos\frac{\pi x}{8}|_{-4}^{4} - \frac{8}{\pi}\int_{-4}^{4} \cos\frac{\pi x}{8}\, dx \\ &= 16 - \frac{8 \times 16}{\pi^2} = 16\left(1 - \frac{8}{\pi^2}\right) \end{aligned}$$

(c) The variance of X is

$$Var(X) = E[X^2] - E^2(X) = 16\left(1 - \frac{8}{\pi^2}\right)$$

6. **Problem 6 Solution:**

(a) The mean of X is

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^1 x \cdot \frac{5}{4}(1-x^4)\,dx \\ &= \frac{5}{4}\left[\int_0^1 x\,dx - \int_0^1 x^5\,dx\right] = \frac{5}{4}\left[\frac{1}{2}x^2|_0^1 - \frac{1}{6}x^6|_0^1\right] = \frac{5}{12} \end{aligned}$$

(b)

$$\begin{aligned} E[4X+2] &= \int_0^1 (4x+2)\cdot\frac{5}{4}(1-x^4)\,dx = \int_0^1 \frac{5}{2}\,(1+2x-x^4-2x^5)dx \\ &= \frac{5}{2}\left(1+1-\frac{1}{5}-\frac{2}{6}\right) = \frac{11}{3} \end{aligned}$$

(c) The second moment is

$$\begin{aligned} E[X^2] &= \int_0^1 x^2\cdot\frac{5}{4}(1-x^4)\,dx = \int_0^1 \frac{5}{4}\,(x^2-x^6)\,dx \\ &= \frac{5}{4}\left(\frac{1}{3}-\frac{1}{7}\right) = \frac{5}{21} \end{aligned}$$

7. **Problem 7 Solution:** X has a Binomial distribution with parameters $(n=3, p=0.7)$. The pmf is give as:

$$\begin{aligned} P(X=0) &= \binom{3}{0} 0.7^0\cdot(1-0.7)^3 = 0.3^3 = 0.027 \\ P(X=1) &= \binom{3}{1} 0.7^1\cdot(1-0.7)^2 = 3\cdot 0.7\cdot 0.3^2 = 0.189 \\ P(X=2) &= \binom{3}{2} 0.7^2\cdot(1-0.7)^1 = 3\cdot 0.7^2\cdot 0.3 = 0.441 \\ P(X=3) &= \binom{3}{3} 0.7^3\cdot(1-0.7)^0 = 1\cdot 0.7^3 = 0.343 \end{aligned}$$

8. **Problem 8 Solution:** The mass function is give as

$$P(b) = 0.5\delta(b) + 0.1\delta(b-1) + 0.2\delta(b-2) + 0.1\delta(b-3) + 0.1\delta(b-3.5)$$

where the non-zero mass function is obtained by taking the difference of jump points in CDF curve. (The pmf sketch is ignored.)

9. **Problem 9 Solution:** Let X be the number of correct answers. There are 5 questions, we can think there are 5 trials. Each trial has a probability of 1/3 success and 2/3 fail. Therefore, X is a binomial random variable. The probability of four or more correct answer is

$$\begin{aligned} P(X=4)+P(X=5) &= \binom{5}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^1 + \binom{5}{5}\left(\frac{1}{3}\right)^5 \qquad (1) \\ &= 5\cdot\left(\frac{1}{3}\right)^4\cdot\left(\frac{2}{3}\right) + \left(\frac{1}{3}\right)^5 \end{aligned}$$

10. **Problem 10 Solution:** X is poisson r.v. so

$$P(X=k)=e^{-\lambda}\frac{\lambda^k}{k!}\quad k=0,1,2,\cdots$$

we have
$$\frac{P(X=i)}{P(X=i-1)}=\frac{e^{-\lambda}\frac{\lambda^i}{i!}}{e^{-\lambda}\frac{\lambda^{i-1}}{(i-1)!}}=\frac{\lambda(i-1)!}{i!}=\frac{\lambda}{i}\quad(i=1,2,\cdots)$$

when $i<\lambda$, $P(X=i)>P(X=i-1)$, the pmf is an increasing function. When $i>\lambda$, $P(X=i)<P(X=i-1)$, the pmf is an decreasing function. Therefore, the pmf of X first increases until i reaches the largest integer below λ. Then it decreases monotonically.

11. **Problem 11 Solution:**

(a) Let $Y=cX$, then $\mu_Y=c\cdot\mu_X$, and

$$\begin{aligned}Var(cX) &= Var(Y)=E[(Y-\mu_Y)^2]=E[Y^2-2\mu_Y\cdot Y+\mu_Y^2] \\ &= E[Y^2]-\mu_Y^2=E[c^2X^2]-c^2\mu_X^2=c^2\{E[X^2]-\mu_X^2\} \\ &= c^2\cdot Var(X)\end{aligned}\qquad(2)$$

(b) Let $Y=c+X$, then $\mu_Y=c+\mu_X$,

$$\begin{aligned}Var(Y) &= E[(Y-\mu_Y)^2]=E[(c+X-c-\mu_X)^2] \\ &= E[(X-\mu_X)^2]=Var(X)\end{aligned}\qquad(3)$$

12. **Problem 12 Solution:** The mass function of discrete RV X is

$$P_X(x=n)=\begin{cases}\frac{1}{3} & (n=1,2,3)\\ 0 & (o.w.)\end{cases}$$

First, we can find moment by direct derivation:

$$\begin{aligned}E[X] &= \frac{1}{3}\cdot1+\frac{1}{3}\cdot2+\frac{1}{3}\cdot3=\frac{1}{3}\cdot6=2 \\ E[X^2] &= \frac{1}{3}\cdot1+\frac{1}{3}\cdot2^2+\frac{1}{3}\cdot3^2=\frac{1}{3}\cdot(1+4+9)=\frac{14}{3} \\ E[X^3] &= \frac{1}{3}\cdot1+\frac{1}{3}\cdot2^3+\frac{1}{3}\cdot3^3=\frac{1}{3}\cdot(1+8+27)=12\end{aligned}$$

while moment generation function of X is

$$\Phi(\omega)=E[e^{\omega X}]=\sum_i e^{\omega x_i}\cdot P(X=x_i)=\frac{1}{3}\left(e^{\omega}+e^{2\omega}+e^{3\omega}\right)$$

The first derivative of $\Phi(\omega)$ is

$$\Phi'(\omega)=\frac{1}{3}(e^{\omega}+2e^{2\omega}+3e^{3\omega})$$

Hence, $E[X]$ can be obtained as

$$E[X] = \Phi'(\omega)|_{\omega=0} = \frac{1}{3}(1+2+3) = 2$$

Similarly,

$$\Phi''(\omega) = \frac{1}{3}(e^{\omega} + 2^2 e^{2\omega} + 3^2 e^{3\omega})$$

and

$$\begin{aligned} E[X^2] &= \Phi''(\omega)|_{\omega=0} = \frac{1}{3}(1+2^2+3^2) = \frac{14}{3} \\ E[X^3] &= \Phi'''(\omega)|_{\omega=0} = \frac{1}{3}(1+2^3+3^3) = 12 \end{aligned}$$