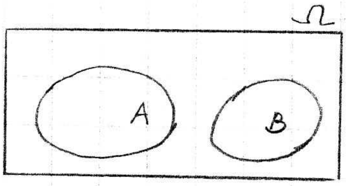
# Solution to Assignment I.

1. method 1.




Since $AB = \phi$, we must have $A \subset B^c$. Thus, $P(A) \leq P(B^c)$

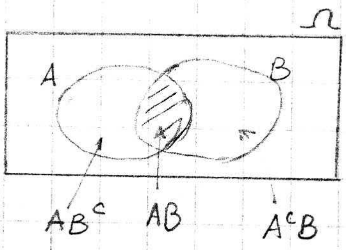
method 2:

since $AB = \phi$,

$P(A \cup B) = P(A) + P(B) \leq 1$

$\therefore \; P(A) \leq 1 - P(B)$ which implies $P(A) \leq P(B^c)$

2.




By total probability, we have

$P(A) = P(AB) + P(AB^c)$ and $P(B) = P(AB) + P(A^cB)$

We are given $P(A) = P(B) = P(AB)$, so that

$P(AB^c) = 0$ and $P(A^cB) = 0$

Then

$$P(AB^c \cup A^cB) = P(AB^c) + P(A^cB) = 0$$

3.

$P(A \cup B) = P(A) + P(B) - P(AB)$

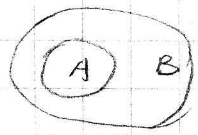
$\therefore \; P(A \cup B) + P(AB) = P(A) + P(B) = 2$

while $0 \leq P(A \cup B) \leq 1$ and $0 \leq P(AB) \leq 1$

gives $P(A \cup B) = P(AB) = 1$

4. $A \subset B \Rightarrow P(AB) = P(A)$

$A \subset B$




$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(A)}{P(B)} = \frac{1/4}{1/3} = \frac{3}{4}$$

$$P(B|A) = \frac{P(AB)}{P(A)} = \frac{P(A)}{P(A)} = 1$$

5. Define the events:

$R \equiv \{\text{red ball drawn}\}$.

$B_1 \equiv \{\text{box 1 was selected}\}$. we need to find $P(B_1|R)$.

and we have the probabilities

$P(B_1) = P(B_1^c) = \frac{1}{2}$.

$P(R|B_1) = \frac{999}{1000}$, $P(R|B_1^c) = \frac{1}{1000}$

we can apply Bayes theorem to get

$$P(B_1|R) = \frac{P(R|B_1)\,P(B_1)}{P(R|B_1)\,P(B_1) + P(R|B_2)\,P(B_2)} = \frac{999/1000 \cdot \frac{1}{2}}{999/1000 \cdot \frac{1}{2} + 1/1000 \cdot 1/2}$$

$$= 999/1000$$

6. Define the events

$D \equiv \{\text{both parts are defective}\}$.

$A \equiv \{\text{parts come from bin A}\}$.

So $P(D|A) = \frac{100}{1000} \cdot \frac{99}{999}$ and $P(D|A^c) = \frac{100}{2000} \cdot \frac{99}{1999}$

and $P(A) = P(A^c) = \frac{1}{2}$.

(a). $P(D) = P(D|A)\,P(A) + P(D|A^c)\,P(A^c)$

$$= \frac{100}{1000} \cdot \frac{99}{999} \cdot \frac{1}{2} + \frac{100}{2000} \cdot \frac{99}{1999} \cdot \frac{1}{2} \approx 0.006$$

(b) $P(A|D) = \dfrac{P(D|A)\,P(A)}{P(D)} \doteq 0.8$

7. Define the events

$Y \equiv \{$ you picked the prize door $\}$

$O \equiv \{$ 'other' door is the prize door $\}$

since you pick without knowledge. $P(Y) = \frac{1}{3}$, $P(Y^c) = \frac{2}{3}$

The person running the contest has perfect knowledge, and reveals a non-prize door conditioned on your choice, so that

$$P(O|Y^c) = 1 \quad \text{and} \quad P(O|Y) = 0$$

Thus
$$P(O) = P(O|Y)\,P(Y) + P(O|Y^c)\,P(Y^c)$$
$$= 0 \cdot \frac{1}{3} + 1 \cdot \frac{2}{3} = \frac{2}{3}$$

If you have trouble seeing that, consider a similar contest where you pick one of 1000 doors and the person running the contest reveals no prize behind 998 of the remaining 999 doors. With probability 0.999 you did not pick the prize door, and it is the remaining one of the others.

8. We are given $X \sim N(\mu, \sigma^2)$ with $\mu = 0$ and $\sigma = 2$, so

$$f_X(x) = \frac{1}{2\sqrt{2\pi}}\, e^{-\frac{x^2}{2\cdot 2^2}} = \frac{1}{\sqrt{8\pi}}\, e^{-\frac{x^2}{8}} \quad \text{for } x \in \mathbb{R}$$

(a) $P_X(\{x \in \mathbb{R}, 1 \le x \le 2\}) = \Phi\left(\frac{2-\mu}{\sigma}\right) - \Phi\left(\frac{1-\mu}{\sigma}\right)$

$$= \Phi(1) - \Phi(0.5) \approx 0.1498$$

(b) $P_{X|X\geq 1}\left(\{x\in R : 1\leq x\leq 2\}\right)$

$$= \frac{P_X\left(\{x\in R: x\geq 1\}\cap\{x\in R: 1\leq x\leq 2\}\right)}{P_X\left(\{x\in R: x\geq 1\}\right)}$$

$$= \frac{P_X\left(\{x\in R: 1\leq x\leq 2\}\right)}{P_X\left(\{x\in R: r\geq 1\}\right)}$$

$$= \frac{\phi\left(\frac{2-\mu}{\sigma}\right)-\phi\left(\frac{1-\mu}{\sigma}\right)}{\phi\left(\frac{\infty-\mu}{\sigma}\right)-\phi\left(\frac{1-\mu}{\sigma}\right)} = \frac{\phi(1)-\phi(0.5)}{1-\phi(0.5)} \approx 0.486$$

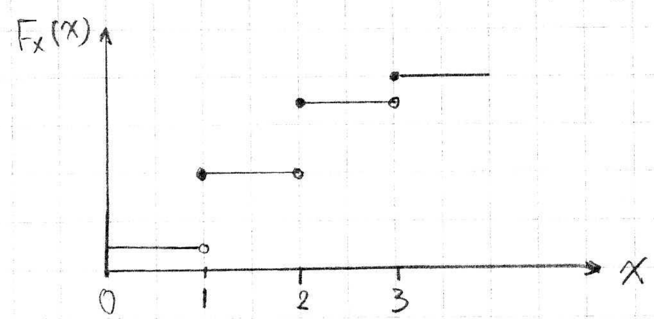
Here $\phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt$ is the standard normal cdf.

9. Clearly, X is a discrete random variable with pmf

$$P_X(k) = \binom{3}{k}\left(\tfrac{1}{2}\right)^k\left(\tfrac{1}{2}\right)^{3-k} \quad \text{for } k=0,1,2,3$$

$$= \begin{cases} 1/8 & k=0,3 \\ 3/8 & k=1,2 \end{cases}$$

Thus, $F_X(x) = \begin{cases} 0 & x<0 \\ 1/8 & 0\leq x<1 \\ 4/8 & 1\leq x<2 \\ 7/8 & 2\leq x<3 \\ 1 & x\geq 3 \end{cases}$

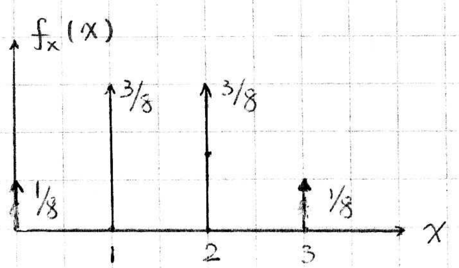


There are discontinuities at $x = 0, 1, 2, 3$ due to the discrete probabilities there. We can use delta functions to form a pdf. Thus

$$f_X(x) = \frac{d}{dx} F_X(x) = \begin{cases} 0 & x \neq 0, 1, 2, 3 \\ 1/8\ \delta(x) & x = 0 \\ 3/8\ \delta(x-1) & x = 1 \\ 3/8\ \delta(x-2) & x = 2 \\ 1/8\ \delta(x-3) & x = 3 \end{cases}$$

or $f_X(x) = \frac{1}{8}\delta(x) + \frac{3}{8}\delta(x-1) + \frac{3}{8}\delta(x-2) + \frac{1}{8}\delta(x-3)$

for $x \in R$




10. Define a random variable

$X$ = arrival time of professor in hours past 8:00 am.

So $f_X(x) = 1$ for $0 \leq x \leq 1$

and the events can be restated as

$$A = \{x \in R : \tfrac{1}{2} < x \leq 1\}.$$

$$B = \{x \in R : 0 \leq x \leq 31/60\}.$$

(a) $$P(B \mid A) = \frac{P(AB)}{P(A)} = \frac{P(\{x \in R : 1/2 < x \leq 31/60\})}{P(A)}$$

$$= \frac{\int_{1/2}^{31/60} f_X(x)\,dx}{\int_{1/2}^{1} f_X(x)\,dx} = \frac{1/60}{1/2} = \frac{1}{30}$$

(b) $$P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(\{x \in R : \frac{1}{2} < r \leq \frac{31}{60}\})}{P(B)}$$

$$= \int_{1/2}^{31/60} f_X(x)\,dx \Big/ \int_0^{31/60} f_X(x)\,dx$$

$$= \frac{1}{60} \Big/ \frac{31}{60} = \frac{1}{31}$$

11 (a) Using $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$, we have

$$1 = \int_{-\infty}^{\infty} c e^{-2x}\,dx = \left[-\frac{c}{2} e^{-2x}\right]_0^{\infty} = \frac{c}{2}$$

$\therefore\ c = 2$

(b) For $x > 0$ and $a > 0$, we have

$$P_X(\{r \in R : r \geq x + a\})$$

$$= \int_{x+a}^{\infty} 2 \cdot e^{-2z}\,dz = \left[-e^{-2z}\right]_{x+a}^{\infty} = e^{-2(x+a)}$$

(c) $$P_{X|X \geq a}(\{r \in R : r \geq x + a\})$$

$$= \frac{P_X(\{r \in R : r \geq x+a\} \cap \{r \in R : r \geq a\})}{P_X(\{r \in R : r \geq a\})}$$

$$= P_X(\{r \in R : r \geq x+a\}) \Big/ P_X(\{r \in R : r \geq a\})$$

$$= e^{-2(x+a)} \Big/ e^{-2a}$$

$$= e^{-2x}$$ (Note, ~~this~~ does not depend on $a$)