EE8103 (Random Processes) Midterm Examination

Closed-book, 3 hours, Fall 2011

Name: ________________________ Student ID: ______________________

(Total: 6 questions, 35 marks)

1. (4 marks) Assume that each child who is born is equally likely to be a boy or a girl. If a family has two children, a) what is the probability that at least one child is a girl? b) what is the probability that both are girls given that at least one is a girl?

Define: A: at least one child is a girl

B: both are girls.

a) $P(A) = 1 - P(\bar{A}) = 1 - \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}$

b) $P(B|A) = \frac{P(BA)}{P(A)} = \frac{P(B)}{P(A)} = \frac{\frac{1}{2}\times\frac{1}{2}}{\frac{3}{4}} = \frac{1}{3}$

2. (5 marks) The lifetime of a type of light bulb is a random variable with mean 100 hours and variance of 25 hours. A light bulb is used until it fails, at which point it is replaced by a new light bulb. The lifetime of each light bulb is independent, identically distributed (I.I.D). Let random variable Y denote the total lifetime of the 36 light bulbs which are used one by one.

Find: (a) the probability $P[3500 \leq Y \leq 3700]$, and b) the conditional probability $P[Y \leq 3700 \mid Y \geq 3500]$. (Using standard normal CDF $\Phi(x)$ where $x \geq 0$ to denote your results)

$Y \sim N(3600, 36 \times 25)$

a) $P[3500 \leq Y \leq 3700] = \Phi\left(\frac{3700-3600}{30}\right) - \Phi\left(\frac{3500-3600}{30}\right)$

$= 2\Phi\left(\frac{10}{3}\right) - 1$

b) $P[Y \leq 3700 | Y \geq 3500] = \frac{P[3500 \leq Y \leq 3700]}{P[Y \geq 3500]} = \frac{2\Phi(\frac{10}{3}) - 1}{\Phi(\frac{10}{3})}$

3. (6 marks) Random variables X and Y have joint Probability Mass Function (PMF) as below:

$$P_{X,Y}(x,y) = \begin{cases} c(x+y)^2 & x = 0,1; \quad y = 1,2 \\ 0 & otherwise. \end{cases}$$

Find: a) the value of the constant c, b) the probability $P[Y \geq X]$, c) the marginal PMFs $P_X(x)$ and $P_Y(y)$, and d) the covariance $Cov(X,Y)$.

X \ Y	1	2
0	c	4c
1	4c	9c

a) $c + 4c + 4c + 9c = 1$

$\therefore c = \frac{1}{18}$

b) $P[Y \geq X] = 1$

c) $P_X(x) = \begin{cases} \frac{5}{18} & x=0 \\ \frac{13}{18} & x=1 \\ 0 & o.w \end{cases}$ $\qquad P_Y(y) = \begin{cases} \frac{5}{18} & y=1 \\ \frac{13}{18} & y=2 \\ 0 & o.w \end{cases}$

d) $Cov(X,Y) = E(XY) - E(X)E(Y)$

$E(X) = \frac{13}{18}$, $\quad E(Y) = \frac{31}{18}$

$E(XY) = \frac{22}{18}$. $\qquad Cov(X,Y) = -0.0216$

4. (6 marks) Random variable Y has the characteristic function (CF) $\Phi_Y(\omega) = 1/(1-j\omega)$. Random variable V has the CF $\Phi_V(\omega) = 1/(1-j\omega)^4$. Y and V are independent. Random variable W is given by $W = Y + V$.

Find: a) $E[Y]$, b) $Var[Y]$, and c) the characteristic function of random variable W: $\Phi_W(\omega)$.

a) $E(Y) = \frac{1}{j} \frac{\partial \phi_Y(\omega)}{\partial \omega}\Big|_{\omega=0} = 1$

b) $E(Y^2) = \frac{1}{j^2} \frac{\partial^2 \phi_Y(\omega)}{\partial \omega^2}\Big|_{\omega=0} = \frac{2}{(1-j\omega)^3}\Big|_{\omega=0} = 2$

$Var(Y) = E(Y^2) - (E(Y))^2 = 1$

c) $\Phi_W(\omega) = \Phi_Y(\omega)\,\Phi_V(\omega) = \frac{1}{(1-j\omega)^5}$

5. (7 marks) Random variable X has a Probability Density Function (PDF)

$$f_X(x) = \begin{cases} 3x^2 & 0 \le x \le 1 \\ 0 & otherwise \end{cases}$$

Random variable Y is given by $Y = \begin{cases} 2X + 1 & 0 \le X < 0.5 \\ 4X & 0.5 \le X \le 1 \end{cases}$

Find: a) the CDF of X: $F_X(x)$, b) the CDF of Y: $F_Y(y)$, and c) the PDF of Y: $f_Y(y)$.

a) $F_X(x) = \begin{cases} 0 & x<0 \\ x^3 & 0 \le x \le 1 \\ 1 & x>1 \end{cases}$

b) $F_Y(y) = \begin{cases} 0 & y<1 \\ \left(\frac{y-1}{2}\right)^3 & 1 \le y < 2 \\ \left(\frac{y}{4}\right)^3 & 2 \le y \le 4 \\ 1 & y>4 \end{cases}$

c) $f_Y(y) = \begin{cases} \frac{3}{2}\left(\frac{y-1}{2}\right)^2 & 1 \le y < 2 \\ \frac{3}{4}\left(\frac{y}{4}\right)^2 & 2 \le y \le 4 \\ 0 & o.w \end{cases}$

6. (7 marks) Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} c(2x + y) & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & otherwise \end{cases}$$

Find: a) the constant c, b) the marginal PDF $f_X(x)$, and c) $P[\min(X, Y) < 0.5]$.

a) $\int_0^1\int_0^1 c(2x+y)\,dx\,dy = 1$

$\therefore c = \frac{2}{3}$

b) $f_X(x) = \int_0^1 \frac{2}{3}(2x+y)\,dy = \frac{4}{3}x + \frac{1}{3}, \quad 0 \le x \le 1$

$f_X(x) = \begin{cases} \frac{4}{3}x + \frac{1}{3} & 0 \le x \le 1 \\ 0 & o.w \end{cases}$

c) $P[\min(X,Y) < 0.5] = 1 - P[\max(X,Y) \geq 0.5]$

$= 1 - \int^1 \int^1 \frac{2}{3}(2x+y)\,dx\,dy = \frac{5}{8}$